Ordered Context Effects

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Abstract

This paper proposes a unified framework of context-dependent preferences where departures from structural/parametric models are explained by an ordered set of preference parameters. A meta-framework captures context effects through endogenous reference formation and introduces novel models of risk, time, and social preferences. When reference points are fixed, canonical models like expected utility and exponential discounting hold; otherwise, "non-standard" behaviors arise from monotonic changes in preference parameters (e.g., CRRA coefficients, discount factors) along a reference order. Conceptually, the framework links structural violations to basic rationality violations, even though these assumptions are conventionally regarded as independent building blocks in individual decision-making.

Keywords: Context effects, ordered utility, bounded rationality, structural assumptions, reference dependence
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1 Introduction

The standard models of individual decision-making face two distinct strands of empirical challenges. First, choices over generic alternatives violate the basic premise of utility maximization due to the failure of *basic rationality* like the Weak Axiom of Revealed Preference (WARP) and the Independence of Irrelevant Alternatives. Second, in richer domains such as risky choice and delayed payments, *structural assumptions* like expected utility and exponential discounting are challenged by well-known "anomalies" that include, among others, the Allais paradox and present-biased behavior. These two classes of departures have largely been studied separately, and independently for each domain of choice, leading to the development of models that explain one phenomenon in isolation of the others.[1]

This paper takes a different approach by studying how the two classes of violations can be inherently related to one another in a unified and tractable framework of context-dependent preferences. Novel models in risk, time, and social preferences preserve canonical functional forms but allow preference parameters to change monotonically across choice sets, formally capturing how structural violations—behaviors that appear to violate well-known functional forms—may ultimately be rooted in the failures of basic rationality.

To see the main intuition, suppose a decision maker (DM) uses a mental process that is consistent with a "normative" theory but has context-dependent taste that manifests as menu-dependent preference parameters. That is, although the DM uses the same set of preference parameters (e.g., utility functions, discount factors) and canonical functional forms to evaluate all available alternatives in a given problem, as is the case in standard models like expected utility and exponential discounting, her preference parameters can change from one problem to another due to context effects. Then, by looking at the right choice problems, the DM can violate both classes of assumptions—basic rationality assumption (e.g., WARP) and structural assumptions (e.g., Independence, Stationarity, Quasi-linearity)—when the context has changed. However, for a fixed context, which corresponds to a set of choice problems that share the same reference points, canonical functional forms combined with persistent preference parameters lead to behavior that fully complies with both classes of assumptions.

From this perspective, the two classes of assumptions conventionally treated as separate building blocks of decision-making are inextricably linked, and novel behavior violates structural assumptions because, more fundamentally, it cannot be explained by a persistent (non-parametric)

[1] **Cases that maintain basic rationality**: rank-dependent utility (Quiggin, 1982), quadratic utility (Machina, 1982), betweenness preferences (Chew, 1983; Fishburn, 1983; Dekel, 1986), disappointment aversion (Gul, 1991), quasi-hyperbolic discounting (Phelps and Pollak, 1968; Laibson, 1997), hyperbolic discounting (Loewenstein and Prelec, 1992; Frederick, Loewenstein, and O'donoghue, 2002), reference-dependent utility with loss aversion (Kőszegi and Rabin, 2007), cautious expected utility (Cerreia-Vioglio, Dillenberger, and Ortoleva, 2015), general notions of decreasing impatience (Chakraborty, 2021; Chambers, Echenique, and Miller, 2023). **Other cases**: Dubra, Maccheroni, and Ok (2004); Hara, Ok, and Riella (2019) maintain Independence but relax basic rationality. **Beyond the standard primitive**: Ortoleva (2010); Kovach (2020) consider a system of reference-dependent preference relations and study structural violations, Bordalo, Gennaioli, and Shleifer (2012); Lanzani (2022) consider correlated lotteries where they relax (state-independent) Independence and (state-independent) Transitivity, Noor and Takeoka (2015) consider two-stage self-control problems where they relax Independence (for ex-ante preference) and WARP (for ex-post choices).

utility function. The framework naturally applies to well-known "anomalies" in the risk, time, and social domains, which have predominantly been analyzed under the assumption of basic rationality throughout decades of decision theory.

Risk Domain. Empirically documented "anomalies" in risky choices suggest that people exhibit increased risk aversion when safer options become available (Allais, 1990; Wakker and Deneffe, 1996; Herne, 1999; Bleichrodt and Schmidt, 2002; Andreoni and Sprenger, 2011). While this behavior contradicts the expected utility theory, it may align with the expected utility functionals when coupled with context-dependent utility functions that vary in concavity. This observation motivates the *Ordered Expected Utility* model in Section 3, where the choice from menu A is given by

$$c(A) = \arg\max_{p \in A} \sum_{x} p(x)\, u_{r(A)}(x). \tag{1.1}$$

Standard expected utility applies when the safest alternative, which acts endogenously as the reference r, is the same across a collection of choice sets, i.e., $r(A_1) = r(A_2) = ... = r(A_n)$. In this case, the same Bernoulli utility function $\bar{u} = u_{r(A_1)} = u_{r(A_2)} = ... = u_{r(A_n)}$ is used to evaluate all alternatives in the choice sets $A_1, A_2, ...A_n$, and the DM appears to be completely standard. But when the safest alternative changes from one choice set to another, a safer reference leads to a more concave utility function u_r, which gives rise to increased risk aversion. Since expected utility functionals apply, this form of context-dependent preferences can be tractably modeled using CRRA utility functions where the risk aversion coefficient is determined by the variance of the safest available option.

Time Domain. Well-documented evidence of present bias suggests that individuals are less patient in short-term decisions, a behavior at odds with exponential discounting (Laibson, 1997; Frederick, Loewenstein, and O'donoghue, 2002; Benhabib, Bisin, and Schotter, 2010; Chakraborty, 2021). However, this behavior could be consistent with the exponential discounting form when paired with context-dependent discount factors that capture changing time preferences, motivating the *Ordered Discounted Utility* model in Section 4.

$$c(A) = \arg\max_{(x,t) \in A} \delta_{r(A)}^{t} u(x) \tag{1.2}$$

When a problem offers sooner payments, it alters the reference point r, prompting the DM to use a lower discount factor δ_r that reflects increased impatience. Again, changes in preferences are systematic along a certain order and behavior is completely standard for menus that share the same soonest payment.

Social Domain. Empirical evidence in economics and psychology suggests that the same individual can display different degrees of altruism in different choice settings, for example when a

balanced split of reward is available than when it is not (Ainslie, 1992; Rabin, 1993; Nelson, 2002; Fehr and Schmidt, 2006; Sutter, 2007). This form of context-dependent preference motivates the *Ordered Social Utility* model in Section 5, where a DM choosing how much to keep (x) and give (y) uses a mental process consistent with quasi-linear utility but with utility from giving $v_{r(A)}$ that is menu-dependent.

$$c(A) = \arg\max_{(x,y)\in A} x + v_{r(A)}(y) \tag{1.3}$$

When a more-equitable split becomes available, changing the reference from r' to r'', increased altruism is captured by the supermodularity condition: $v_{r''}(y_2) - v_{r''}(y_1) > v_{r'}(y_2) - v_{r'}(y_1)$ for all $y_2 > y_1$.

Unified Framework. The models, motivated by different non-standard behaviors, depart from their canonical counterparts (i.e., expected utility, exponential discounting, quasi-linear utility) in a common fashion. First, after a menu realizes, the DM is canonical, and analysis should proceed as usual. It is as-if behavior is non-standard not because normative mental processes fail but because preferences are changing. Second, preference changes are tractable, where contexts are captured by reference points that can be ordered and changes in preferences are monotonic along the reference order.

More formally, the common foundation for all three models, the *Ordered Utility* representation, is introduced in Section 2. A reference order R defined on the space of alternatives is maximized to identify the reference alternative $r(A)$ of a menu A. Then, $r(A)$ determines a general utility function $U_{r(A)}$ that the DM maximizes.

$$c(A) = \arg\max_{z\in A} U_{r(A)}(z) \tag{1.4}$$

Equations 1.1, 1.2, and 1.3 from the risk, time, and social domains are special cases of Equation 1.4. Because the DM maximizes possibly distinct utility functions $U_{r(A)}$ for different menus, basic rationality (the assumption that one persistent utility function is maximized) can fail despite the fact that these models are primarily motivated by structural violations. On the other hand, if the same general utility function $U_{r(A_i)}$ is used for menus $A_1, ..., A_n$, then choices from these menus must also satisfy structural assumptions like expected utility and exponential discounting.

Therefore, the models capture a particular kind of context-dependent behavior where structural assumptions and rationality assumptions are inextricably linked. Examples in Sections 3, 4, and 5 show that, given a structural violation like the Allais paradox or a present bias behavior, it is possible to pinpoint exactly what type of WARP violations would be committed, suggesting a clear way to test these theories. On the other hand, and perhaps unexpectedly at first, Propositions 1, 2, and 3 show that compliance with structural assumptions in binary comparisons implies compliance with WARP, and vice versa. The two types of assumptions are hence linked, and a straightforward corollary is that behavior admits a structural utility representation if and only if it admits any utility representation. Therefore, structural "anomalies" in this framework are fundamentally

rooted in the changes of (non-parametric) preferences rather than being a mere failure of structural assumptions, a plausible connection obscured in studies that assume basic rationality.

Behavioral Characterizations. Axiomatic foundations for the unobserved mental processes resonate with one another through a meta-framework called *Reference Dependence* (RD) formally introduced in Section 6. RD performs a context-dependent generalization of arbitrarily given behavioral postulates by calling for their "local" compliance. Roughly speaking, it converts a "normative" axiom into a "locally normative" analog. For the general *Ordered Utility* representation, RD weakens the basic rationality assumption WARP by requiring that in every finite choice set, at least one alternative preserves WARP among choice behavior from subsets (Axiom 6.1). Intuitively, if the failure of rationality is caused by reference dependence where the reference is one of the available alternatives, then rationality should, at least, hold "locally" among choice sets that share the same reference.

For *Ordered Expected Utility*, *Ordered Discounted Utility*, and *Ordered Social Utility*, RD requires "local" compliance of Independence (Axiom 3.2), Stationarity (Axiom 4.2), and Quasi-linearity (Axiom 5.2) while taking domain-specific contexts into account (i.e., safest alternative, earliest payment, most-balanced income distribution). Perhaps unexpectedly at first, this immediately creates a joint generalization of WARP because, whenever the structural postulate is expected to hold, WARP must also hold as a special case. Once the reference order is established, monotonic changes in preference parameters are intuitively captured by behavior that becomes (weakly) more risk averse (Axiom 3.3), impatient (Axiom 4.3), and altruistic (Axiom 5.3) when menus expand.

Departing from the conventional practice of weakening axioms/assumptions incrementally, this framework exploits an underexplored generalization of structural axioms forbidden by traditional adherence to basic rationality. For instance, the Independence axiom requires that if lottery p is preferred to lottery q, then $p^\alpha s$ must be preferred to $q^\alpha s$.[2] Generalizing Independence relaxes this requirement, but WARP still makes it impossible to discuss how $p^\alpha s$ is preferred to $q^\alpha s$ in *some* choice sets while the opposite may hold in others.[3] Therefore, using choice behavior as the primitive and relaxing WARP are indispensable for a broader but intuitive class of generalizations, which paves the way for a context-dependent implementation of structural axioms that can keep us "closer" to canonical functional forms and provide identification of subjective parameters.

Section 2 introduces the basic setup. Sections 3-5 take the framework to the risk, time, and social domains. Section 6 presents the technical, meta-framework underlying all three domains. Section 7 discusses related literature. Section 8 concludes. Key proofs are in Appendix A. Technical results and omitted proofs are relegated to Online Appendix B.

[2] Lottery $p^\alpha s$ refers to the (compound) lottery generated by mixing lottery p with probability α and lottery s with probability $(1-\alpha)$. Independence says the preference between $p^\alpha s$ and $q^\alpha s$ should be the same as the preference between p and q, since they differ only by a common term.

[3] Similar limitations apply when the primitive is a (complete) preference relation.

2 Basic Setup

Let Y be a separable metric space, endowed with the standard Euclidean metric d_2, that represents the set of all alternatives. Let $\mathcal{A}$ be the set of all finite and nonempty subsets of Y, also called choice sets. The primitive of this paper is a choice correspondence $c : \mathcal{A} \to \mathcal{A}$ where $c(B) \subseteq B$ for all $B \in \mathcal{A}$. I assume throughout the paper that c is continuous:

Axiom (Continuity)**.** *c has a closed graph.*[4]

The risk, time, and social preferences studied in this paper share a common starting point: For any given choice set, the DM is seemingly standard by maximizing a single utility function. But, globally, behavior is non-standard because this function depends on possibly different reference alternatives across different choice sets.[5]

Definition 1. A choice correspondence c admits an Ordered Utility (OU) representation if there exist a linear order (R, Y) and a set of utility functions $\{U_y : Y \to \mathbb{R}\}_{y \in Y}$ such that

$$c(A) = \arg\max_{y \in A} U_{r(A)}(y)$$

and $r(A) = \max(R, A)$ for all A, where c has a closed graph.[6]

Theories of ordered reference can be traced back to *triggered rationality* in Rubinstein and Salant (2006), which coincides with OU when restricted to finite alternatives.[7] Kıbrıs, Masatlioglu, and Suleymanov (2023); Giarlotta, Petralia, and Watson (2023); Kıbrıs, Masatlioglu, and Suleymanov (2024) expand on this trajectory by exploring different characterizations, stochastic choice, and connections to psychological constraints and limited consideration, proposing a myriad of intuitive applications.[8]

Despite its intuitiveness, focusing on the generic choice domain poses caveats. Subjective reference points can be hard to identify, and compounding this issue is the lack of tractable changes in preferences that, in turn, burdens the use of ordered reference in applications.

[4] That is, if $x_n \to_d x$, $A_n \to_H A$, and $x_n \in c(A_n)$ for every $n = 1, 2, ...$ then $x \in c(A)$, where $\to_H$ refers to convergence in the Hausdorff distance $d_H(S_1, S_2) = \max\{\sup_{x \in S_1} \inf_{y \in S_2} d_2(x, y), \sup_{y \in S_2} \inf_{x \in S_1} d_2(x, y)\}$.

[5] This naturally bounds non-standard behavior: When $|Y|$ is finite, there are at most $|Y|$ distinct utility functions, but there are around $2^{|Y|}$ choice sets, and this difference increases exponentially in $|Y|$.

[6] A linear order (R, Y) is a complete, reflexive, transitive, and antisymmetric binary relation R on Y. Per convention, $\max(R, A)$ is $x \in A$ such that xRy for all $y \in A \backslash \{x\}$. The closed graph property is trivial if Y is finite. When Y is a separable metric space, it may be more natural to implement structures on R and $\{U_y : Y \to \mathbb{R}\}_{y \in Y}$ (e.g., R is continuous and U_y changes continuously in y along R) so that c, as a result, has a closed graph. However, this approach also introduces ad-hoc restrictions when the model is later specialized to richer domains, hence OU is written with this generality in place.

[7] Ravid and Steverson (2021) studies the same behavior under a model of bad temptation.

[8] In Kıbrıs, Masatlioglu, and Suleymanov (2023), higher positioned search results are conspicuous to consumers, serving as a reference that influences their final decisions. Kıbrıs, Masatlioglu, and Suleymanov (2024) consider the case of marketing campaigns where a consumer is more likely to recall an advertised product and use it as benchmark to make consumption decisions. Giarlotta, Petralia, and Watson (2023) propose that a salient item on a restaurant's menu, such as the frog's legs dish in Luce and Raiffa's Dinner, can become the reference and influence a consumer's preference through increased confidence.

Sections 3-5 (risk, time, and social preferences) demonstrate that richer domains provide natural remedies to and in fact benefit from the flexibility of this model. First, it significantly expands the interpretation of ordered reference, ranging from being partially subjective (ranking lotteries by riskiness, Section 3) to fully objective (Gini index, Section 5), so as to capture domain-specific contexts. Moreover, the reference order serves as an anchor along which monotonic, domain-specific preference shifts are manifested, such as increasing risk aversion or decreasing patience along the established order. The two components—reference order and reference effect—interact with each other, yielding a framework that captures a highly specific and tractable form of context-dependent preferences.

3 Ordered Expected Utility

Consider a DM whose willingness to take risk is dynamic as it depends on how much of it is avoidable. The safest alternative in a choice set provides a natural measure for the underlying context. Sometimes, we have the option to fully avoid risk by keeping our assets in cash or by buying an insurance policy, and so the safest option is quite safe. But in other situations, such as a job search or a carefully designed lab experiment in which all options involve risk, taking some risk becomes unavoidable. This section introduces a model where a DM is more risk averse when risk is avoidable than when it is not. The model preserves the expected utility functionals but allows for risk aversion parameters to be reference-dependent—with the safest available alternative acting as the reference point—where risk aversion increases when safer options become available.

3.1 Preliminaries and axioms

Consider a finite set of prizes $X \subset \mathbb{R}$, where $|X| > 2$, with the largest and smallest prizes denoted by b and w, respectively. Let $Y = \Delta(X)$ be the set of all probability measures over X, called lotteries. Everything else follows Section 2. Per convention, δ denotes a degenerate lottery and δ_x denotes the degenerate lottery that gives prize $x \in X$. For lotteries p, q and $\alpha \in [0,1]$, let $p^\alpha q$ denote the convex combination $\alpha p \oplus (1-\alpha) q$. We say the pair of lotteries (p', q') is a common mixture of the pair of lotteries (p, q) if there exist $\alpha \in (0,1]$ and lottery s such that $p' = p^\alpha s$ and $q' = q^\alpha s$. For every lottery p, let $p(x)$ denote the probability of prize x. I assume first-order stochastic dominance (FOSD):

Axiom 3.1. *If p first-order stochastically dominates q (where $p \neq q$) and $p \in A$, then $q \notin c(A)$.*

The key axiom introduces a *local* compliance of the von Neumann-Morgenstern Independence condition, starting with a definition that applies Independence selectively.

Definition 2. *c satisfies Independence over $\mathcal{S} \subseteq \mathcal{A}$* if for all $A, B \in \mathcal{S}$ and $\alpha \in (0,1]$, if $p \in c(A)$, $q \in A$, $q^\alpha s \in c(B)$, and $p^\alpha s \in B$, then $p^\alpha s \in c(B)$ and $q \in c(A)$.

$\{\boldsymbol{p}, q, p', q'\}$	$\rightarrow$	q		$\{q, p', q'\}$	$\rightarrow$	q
$\{\boldsymbol{p}, q, p'\}$	$\rightarrow$	q		$\{q, q'\}$	$\rightarrow$	q
$\{\boldsymbol{p}, q, q'\}$	$\rightarrow$	q		$\{q, p'\}$	$\rightarrow$	p'
$\{\boldsymbol{p}, p', q'\}$	$\rightarrow$	q'		$\{p', q'\}$	$\rightarrow$	p'
$\{\boldsymbol{p}, q\}$	$\rightarrow$	q		$\{q\}$	$\rightarrow$	q
$\{\boldsymbol{p}, p'\}$	$\rightarrow$	p		$\{p'\}$	$\rightarrow$	p'
$\{\boldsymbol{p}, q'\}$	$\rightarrow$	q'		$\{q'\}$	$\rightarrow$	q'
$\{\boldsymbol{p}\}$	$\rightarrow$	p				

Table 1: Non-empty subsets of $A = \{p, q, p', q'\}$ partitioned into two components using p (highlighted in bold) as an anchor, followed by the choices from these menus. Independence is violated across components but satisfied within the left component (when p is available).

Standard expected utility requires c to satisfy Independence over the set of all problems $\mathcal{A}$. Axiom 3.2 departs from standard expected utility by allowing for reference-dependent preferences but demands local compliance with the Independence axiom for any collection of choice sets that "share a reference."

To illustrate local compliance, Table 1 presents a choice behavior that fails to satisfy Independence (globally), e.g., q is chosen from $\{p, q\}$ but p' is chosen from common mixtures $\{p', q'\}$. However, the choices organized on the left, all of which from choice sets that contain the potential reference p, satisfy Independence (locally).

Before presenting the axiom formally, an additional requirement narrows down the range of potential reference to those that may be deemed safest by the DM. If p is a mean-preserving spread of q (pMPSq), it is clearly not the safest. A second-order complements MPS by also deeming lotteries with increased probabilities of the most extreme prizes (but keeping the relative probability of intermediate prizes the same) to be riskier. Formally, p is an extreme spread of q (pESq) if $p = \beta q \oplus (1-\beta)(\alpha\delta_b \oplus (1-\alpha)\delta_w)$ for some $\beta \in [0,1)$ and $\alpha \in (q(b), 1-q(w))$.[9]

Definition 3. Let $\Psi(A) := \{p \in A : \text{for all } q \in A, \text{ neither } p\text{MPS}q \text{ nor } p\text{ES}q\}$ be the set of *least risky lotteries* in A.

Axiom 3.2 (Risk Reference Dependence)**.** *For every $A \in \mathcal{A}$, c satisfies Independence over $\{B \in \mathcal{A} : p \in B \subseteq A\}$ for some $p \in \Psi(A)$.*

One last axiom captures changing risk aversion when more options become available. It is

[9]The two risk orders are non-contradictory and typically non-nested. Extreme spread is intuitively related to Aumann and Serrano (2008)'s risk index, where lotteries are deemed safer in the "economics sense"—under standard expected utility, the extreme spreads of q are lotteries in $\text{conv}(\{q, \delta_b, \delta_w\})$ that are preferred to q by a more-risk-loving DM if a more-risk-averse DM does so. Non-contradictory: extreme spreads of q live in $\text{conv}(\{q, \delta_b, \delta_w\})$, which does not contain any mean preserving contraction of q. Non-nested: extreme spreads need not preserve mean, mean preserving spreads need not maintain relative probability of intermediate prizes; in the special case where $|X| \leq 3$, mean preserving spreads are nested in extreme spreads.

standard to say that a preference relation $\succsim_1$ is *more-risk-averse* than another preference relation $\succsim_2$ if, for any degenerate lottery δ and lottery p, $\delta \succsim_2 p$ implies $\delta \succsim_1 p$. This definition is often studied alongside expected utility but is well-defined in its own right. Axiom 3.3 extends this definition to lotteries that differ by a degenerate component: where $p^\alpha s$ can be obtained from $\delta^\alpha s$ by reallocating probabilities from one prize to one or more prizes. Then, it posits that a DM cannot be more-risk-loving when a choice set expands. The underlying intuition is that additional alternatives should only be able to increase the extent to which risk is avoidable, and if the avoidability of risk (weakly) increases risk aversion, then the additions must not result in increased risk tolerance.

Axiom 3.3. *Suppose $B \subset A$ and $(\delta_1, p_1), (\delta_2, p_2)$ are common mixtures of (δ, p). If $\delta_2 \in c(B)$ and $p_2 \in B \backslash c(B)$, then $\delta_1 \in A$ implies $p_1 \notin c(A)$.*

3.2 Model

Definition 4. c admits an Ordered Expected Utility (OEU) representation if it admits an OU representation $\left(\{U_r\}_{r \in Y}, R\right)$ such that for some strictly increasing (Bernoulli) utility functions $\{u_r : X \to \mathbb{R}\}_{r \in Y}$,

- $U_r(p) = \sum_x p(x)\, u_r(x)$,
- pMPSq and pESq each implies qRp,
- qRp implies $u_q = f \circ u_p$ for some concave function f:$\mathbb{R} \to \mathbb{R}$.

Theorem 1. *c satisfies Axioms 3.1-3.3 and Continuity if and only if it admits an OEU representation.*

When choice behavior admits an OEU representation, it is as if a reference alternative $r(A)$ for the problem A is first determined by R, which ranks safer alternatives higher, and then the DM maximizes expected utility using the associated context-dependent (Bernoulli) utility function $u_{r(A)}$. Because the same $u_{r(A)}$ is used to evaluate all alternatives in A, the DM is, as if, using a mental process that is locally consistent with the expected utility theory. Moreover, a safer reference leads to a (weakly) more concave utility function. This generalizes the standard model where a DM maximizes expected utility using a single utility function throughout, but the departure from expected utility is limited to systematic changes in risk attitude. It can be shown that (for a fixed R) each u_r is unique up to positive affine transformation, except possibly when $r = b^\alpha w$.[10]

Parametric example A simple example of OEU is when utility functions exhibit constant relative risk aversion (CRRA),

$$u_\gamma(x) = \frac{x^{1-\gamma} - 1}{1 - \gamma},$$

[10]Uniqueness is demonstrated in the proof of Theorem 1. When $r = b^\alpha w$, it is possible that u_r is only used to evaluate lotteries that first order stochastically dominates / dominated by r, so that any strictly increasing transformation of u_r is acceptable.

where γ falls in a subjective, individual-specific range $\left[\underline{\gamma}, \bar{\gamma}\right]$. For a fixed problem, the DM chooses as though she is maximizing standard expected utility using u_γ for some fixed γ. Her risk aversion γ gradually increases when more options become available. One individual may be moderately but consistently risk averse, corresponding to a very small range of CRRA coefficients, whereas another individual may be occasionally risk neutral but sometimes very risk averse, corresponding to a large range of CRRA coefficients.

In applications where extreme prizes are not involved, the value γ can also be intuitively parameterized using any risk index (of lotteries) that is consistent with the mean-preserving spread, such as the variance.[11] For example, each lottery p may be associated with an index α that is increasing in its variance Var_p,

$$\alpha(p) := \frac{\text{Var}_p}{1 + \kappa \text{Var}_p},$$

the reference alternative for menu A minimizes variance and achieves $\alpha^*(A) := \min_{p \in A} \alpha(p)$, and the CRRA parameter for A is given by

$$\gamma(A) := \alpha^*(A)\,\underline{\gamma} + (1 - \alpha^*(A))\,\bar{\gamma},$$

where $\underline{\gamma}$ $\bar{\gamma}$, and $\kappa > 0$ are individual-specific parameters representing minimum risk aversion, maximum risk aversion, and sensitivity to context, respectively. For fixed $\underline{\gamma}$ and $\bar{\gamma}$, a higher κ leads to a bigger increase in risk aversion for the same amount of decrease in a problem's minimum variance.

Allais under expected utility functionals The common ratio effect in binary comparisons refers to the observation that, say, the sure prize of $\$3000$ (p_1) is preferred to a lottery that yields $\$4000$ with 80% chance (p_2) but a lottery that yields $\$4000$ with 20% chance (q_2) is preferred to a lottery that yields $\$3000$ with 25% chance (q_1). If treated as separate decisions, the former decision corresponds to a (Bernoulli) utility function that is more concave than the latter's under the expected utility functional.[12] Building on this observation, OEU preserves the expected utility functional but allows the DM to use a more concave utility function in $\{p_1, p_2\}$, permitting the observed pair of choices (p_1 and q_2) but rules out the opposite pair (p_2 and q_1).[13] The prediction generalizes to other lotteries.[14]

[11] Extreme prizes (b and w) require additional treatment to comply with the extreme spread requirement in R.

[12] Let $A = \{p_1, p_2\}$ and $B = \{q_1, q_2\}$. Suppose u_A (resp. u_B) explains the choice from A (resp. B) under expected utility. After normalization (for example $u_A(0) = u_B(0) = 0$ and $u_A(4000) = u_B(4000) = 1$), choice pattern (p_1, q_2) arises if and only if $u_A(3000) > 0.8$ and $u_B(3000) < 0.8$, which in turn implies u_A is a concave transformation of u_B.

[13] Continuing from Footnote 12, the opposite behavior requires $u_B(3000) > u_A(3000)$ and is ruled out. This observation coincides with the *Negative Certainty Independence* postulate in Dillenberger (2010); Cerreia-Vioglio, Dillenberger, and Ortoleva (2015).

[14] Consider a degenerate lottery δ and a lottery p such that neither of them first order stochastically dominates another. Consider the lotteries $\delta' = \delta^\alpha q$ and $p' = p^\alpha q$ where q is a lottery and $\alpha \in (0,1)$, and suppose $|X| = 3$. If $\delta \in c(\{\delta, p\})$ and $p' \in c(\{\delta', p'\})$, then for all $u_1, u_2 : X \to \mathbb{R}$ such that u_1 explains the first choice and u_2 explains the second choice, it is straightforward to show that $u_1 = f \circ u_2$ for some concave function $f : \mathbb{R} \to \mathbb{R}$. Moreover, these choices can always be explained by an OEU representation such that $r(\{\delta, p\})\, R r(\{\delta', p'\})$. Conversely, suppose the choices $c(\{\delta, p\})$ and $c(\{\delta', p'\})$ admit an OEU representation such that $r(\{\delta, p\})\, R r(\{\delta', p'\})$, then $p \in c(\{\delta', p'\})$ whenever $p \in c(\{\delta, p\})$

Allais as WARP violations Perhaps because the Allais paradox is a direct failure of the structural assumption Independence, many models seek to explain this "anomaly" by generalizing Independence while maintaining the basic rationality assumption WARP. OEU takes a different approach by linking Independence violations to the failure of WARP. Intuitively, the use of multiple utility functions leads to a novel manifestation of the common ratio effect as WARP violations. Specifically, consider the lotteries $p_1 = \delta_{3000}$, $p_2 = 0.5\delta_{4000} \oplus 0.5\delta_0$, $q_1 = 0.2\delta_{4000} \oplus 0.7\delta_{3000} \oplus 0.1\delta_0$, and $q_2 = 0.4\delta_{4000} \oplus 0.3\delta_{3000} \oplus 0.3\delta_0$ related by common mixture.[15] A DM who chooses p_1 over p_2, q_2 over q_1, and q_1 over p_1 in binary comparisons commits the common ratio effect (between the first two choices), which can be reconciled in OEU by a reference order R that ranks p_1 the highest. Now, consider the choice set $\{p_1, q_1, q_2\}$. Since R ranks p_1 the highest, the same utility function is active for $\{p_1, q_1, q_2\}$, $\{p_1, p_2\}$, and $\{p_1, q_1\}$, and therefore expected utility predicts the choice of q_1 from $\{p_1, q_1, q_2\}$. But this is a WARP violation since q_2 is chosen from $\{q_1, q_2\}$. This simple observation introduces a direct link between structural violations and basic rationality violations.

Other evidence While the Allais paradox takes center stage, the evidence and intuition for increased risk aversion in the presence of safer options are also found in a wide range of studies. In a setting meant to test for the compromise effect, Herne (1999) found that the presence of a safer option results in WARP violations in the direction of greater risk aversion. Wakker and Deneffe (1996) introduces the tradeoff *method* to elicit risk aversion without using a sure prize and found that the estimated utility functions are less concave relative to standard methods that involve sure prizes. Andreoni and Sprenger (2011) found similar effects when the safest option is close enough to certainty. Bleichrodt and Schmidt (2002) and Schneider and Day (2018) study how avoidable risk can increase risk aversion and discuss empirical evidence; the former focuses on binary comparisons, while the latter provides practical functional forms.

Linking structural properties to basic rationality It turns out that in an OEU, WARP and Independence are indispensably relaxed in the sense that adding either will bring us back to standard expected utility, an observation that provides a formal separation between OEU and non-expected utility models that uphold basic rationality. It further suggests that Independence violations in this model are more fundamentally linked to preferences changes. Specifically, if c admits *any* utility representation, then it must also have an expected utility representation.[16] Intuitively, the expected utility functional forces Independence violations to be accommodated using different objective functions (U_r), and the maximization of different objective functions lead to WARP violations. Denote by $\mathcal{D}$ the collection of all doubletons in $\mathcal{A}$.

Proposition 1. *If c admits an OEU representation, then the following are equivalent:*

(and equivalently $\delta \in c(\{\delta, p\})$ whenever $\delta' \in c(\{\delta', p'\})$).

[15] $q_1 = \frac{2}{5}p_1 \oplus \frac{3}{5}s$ and $q_2 = \frac{2}{5}p_2 \oplus \frac{3}{5}s$ where $s = \frac{1}{3}\delta_{4000} \oplus \frac{1}{2}\delta_{3000} \oplus \frac{1}{6}\delta_0$.

[16] As is standard, we say c admits a utility representation if there exists a real valued function $U : Y \to \mathbb{R}$ such that $c(A) = \arg\max_{y \in A} U(y)$ for all $A \in \mathcal{A}$.

1. *c satisfies WARP over $\mathcal{A}$.*

2. *c satisfies Independence over $\mathcal{D}$.*

3. *c admits an expected utility representation.*

4. *c admits a utility representation.*

Identification of reference Standard expected utility behavior is a special case of OEU where utility functions coincide, and therefore it is impossible to always achieve unique identification of R. However, two lotteries that index different utility functions always have a unique R relationship. First, if two choice sets differ only by p, and choices are inconsistent with expected utility maximization, then we identify that p ranks highest in R among the other alternatives in the choice set. It turns out that the converse is also true. As long as p and q index different utility functions, if (without loss of generality) pRq, then we can find choice sets A, B such that $p, q \in A$ and $B = A \backslash \{p\}$ where choices from A and B violate WARP, meaning we revealed pRq.[17]

4 Ordered Discounted Utility

The canonical model for time preference is Discounted Utility, where a DM evaluates each payment-time pair (x, t) using exponential discounting, i.e., $U(x, t) = \delta^t u(x)$. But the Stationarity condition implied by this model is routinely challenged by lab and field subjects who switch their choices between two payments when the decision is made in advance, typically favoring the later option for long-term decisions, an actively studied behavioral phenomenon called *present bias* (Laibson, 1997; Frederick, Loewenstein, and O'donoghue, 2002; Benhabib, Bisin, and Schotter, 2010; Halevy, 2015; Chakraborty, 2021; Chambers, Echenique, and Miller, 2023). This section studies how present bias is related to WARP-violating preference changes. The original axioms in Fishburn and Rubinstein (1982) are imposed only among choice sets that share a reference point, which in this case is the soonest available payment, as it partially captures how early in advance a DM is making the decision.

4.1 Preliminaries and axioms

Let $X = [a, b] \subset \mathbb{R}_{>0}$ be a non-degenerate interval of payments and let $T = [0, \bar{t}] \subset \mathbb{R}_{\geq 0}$ be a non-degenerate interval of time points. Let $Y = X \times T$ be the set of all timed payments, where each option $(x, t) \in X \times T$ is a payment of x that arrives at time t. Everything else follows Section 2. To simplify analysis, I assume the upper bound of payments is large enough so that some payment at time $\bar{t}$ is better than the worst payment at time 0, that is, $(b, \bar{t}) \in c(\{(a, 0), (b, \bar{t})\})$. The first axiom is standard: greater payments and sooner payments are better.

[17] The proof of Proposition 1 contains this observation. Essentially, it relies on a less obvious property implied by the model that guarantees the existence of a full-dimensional subset of lotteries that rank below p and q in R but are better than p and q when they act as the reference points.

Axiom 4.1.

1. *If* $x > y$*, then* $c(\{(x,t),(y,t)\}) = \{(x,t)\}$.

2. *If* $t < s$*, then* $c(\{(x,t),(x,s)\}) = \{(x,t)\}$.

The well-known Stationarity condition posits that a DM's preference between two future payments is consistent regardless of when the decision is made. Consider the following definition that allows for selective application.

Definition 5. *c satisfies Stationarity over $\mathcal{S} \subseteq \mathcal{A}$* if for all $A, B \in \mathcal{S}$ and $a \geq 0$, if $(x,t) \in c(A)$, $(y,q) \in A$, $(y, q+a) \in c(B)$, and $(x, t+a) \in B$, then $(x, t+a) \in c(B)$.

Whereas global compliance with Stationarity is captured by $\mathcal{S} = \mathcal{A}$, the next axiom demands local compliance between two choice sets that share an earliest payment.

Definition 6. Let $\Psi(A) := \{(x,t) \in A : t \leq q \text{ for all } (y,q) \in A\}$ be the set of *earliest payments* in A.

Axiom 4.2 (Time Reference Dependence)**.** *If $\Psi(A) \cap \Psi(B) \neq \emptyset$, then c satisfies Stationary over $\{A, B\}$.*

It turns out that Axiom 4.2 is a variant of Axiom 3.2, except that Ψ and the behavioral property being generalized have been adapted to the new domain. More generally, it is an application of the meta-framework RD introduced later in Section 6:

Lemma 1. *c satisfies Axiom 4.2 if and only if for every $A \in \mathcal{A}$ and $(x,t) \in \Psi(A)$, c satisfies Stationarity over $\{B \in \mathcal{A} : (x,t) \in B \subseteq A\}$.*

The next postulate rules out increased patience when more options become available. Intuitively, additional options can only tempt the DM to be impatient, and if an impatient choice is already made from B, for example if (x_1, t_1) is (strictly) chosen over (x_2, t_2) where $t_1 < t_2$, then there is no superset $A \supset B$ such that the DM becomes more patient by choosing $(x_2, t_2 + d)$ in the presence of $(x_1, t_1 + d)$.

Axiom 4.3. *Suppose $B \subset A$, $t_1 < t_2$, and $d \in \mathbb{R}$. If $(x_1, t_1) \in c(B)$ and $(x_2, t_2) \in B \backslash c(B)$, then $(x_1, t_1 + d) \in A$ implies $(x_2, t_2 + d) \notin c(A)$.*

However, this falls short of definitively capturing changes in patience. Even in a completely standard world where every individual maximizes exponentially discounted utility, behavioral differences in delay aversion (among individuals) cannot be definitively decomposed into differences in discounting and differences in consumption utility, an issue discussed in Ok and Benoît (2007). Meaning an individual who prefers the sooner alternative could have *greater* patience paired with lower marginal utility for money.

The next postulate addresses this issues by capturing fixed consumption utilities under varying discounting/patience: Suppose a DM is indifferent between all options in the choice set

$\{(x_1,t_1),(x_2,t_2),(x_3,t_3)\}$, where $x_1 < x_2 < x_3$ and $t_1 < t_2 < t_3$. Then in the choice set $\{(x_1,\lambda t_1),(x_2,\lambda t_2),(x_3,\lambda t_3)\}$ where $0<\lambda<1$, since the delays between options have shortened, a standard exponential discounting DM would pick $(x_3,\lambda t_3)$ as the new choice. Our DM, instead, will face competing forces. On the one hand, the possibility of sooner consumption makes her more impatient; on the other hand, shorter delays between options make later payments more attractive. Allowing her the freedom to resolve these competing forces, the next postulate requires that if she ends up choosing both $(x_1,\lambda t_1)$ and $(x_3,\lambda t_3)$—as if the competing forces are balanced—then she must also choose the intermediate option $(x_2,\lambda t_2)$. The same requirement applies when a common delay (or advancement) d is additionally imposed. Both Axiom 4.3 and Axiom 4.4 are trivially satisfied in exponential discounting.

Axiom 4.4. *Consider* $A = \{(x_1,t_1),(x_2,t_2),(x_3,t_3)\}$ *such that* $t_1 < t_2 < t_3$ *and* $A' = \{(x_1,\lambda t_1+d),(x_2,\lambda t_2+d),(x_3,\lambda t_3+d)\}$ *such that* $0<\lambda<1$ *and* $d\in\mathbb{R}$. *If* $c(A)=A$, *then either* $c(A')=\{(x_1,\lambda t_1+d)\}$, $c(A')=\{(x_3,\lambda t_3+d)\}$, *or* $c(A')=A'$.

4.2 Model

Since we consider the standard environment where sooner is always better, discount factors are restricted to non-negative real numbers strictly less than 1, with the exception of $r=(x,\bar{t})$ for which $\delta_r=1$ is possible.

Definition 7. c admits an Ordered Discounted Utility (ODU) representation if it admits an OU representation $\left(\{U_r\}_{r\in Y},R\right)$ such that for some strictly increasing function $u:X\to\mathbb{R}$ and discount factors $\{\delta_r\}_{r\in Y}$,

- $U_r(x,t)=\delta_r^t u(x)$,
- $t<t'$ implies $(x,t)\,R\,(x',t')$ and $\delta_{(x,t)}\leq\delta_{(y,t')}$,
- $t=t'$ implies $\delta_{(x,t)}=\delta_{(y,t)}$.

Theorem 2. *c satisfies Axioms 4.1-4.4 and Continuity if and only if it admits an ODU representation.*

In this model, it is as if in every instance of decision making, the DM uses a mental process that is consistent with exponentially discounted utility, except that she has context-dependent discount factors. The same discount factor is used to evaluate all alternatives in a problem, but the discount factor depends on the timing of the earliest available payment in the underlying problem. When it is possible to choose an early payment, the DM uses a lower discount factor, resulting in behavior that reflects reduced patience. The model thus delivers present bias behavior using familiar technologies—since the exponential discounting form is preserved in every instance of decision-making, changes in patience are simply captured by set-dependent discount factors. Intuitively, with the entire set of possible payments progressively postponed, the DM begins to treat them more akin to long-term concerns than before, resulting in increased patience.

It can be shown that δ_r is unique given u, except possibly when $r = (x, \bar{t})$.[18] In applications, since the reference order and the discount factors depend only on the timing of a payment, it is without loss to consider discount factors that are based on time rather than on alternatives. This is achieved by setting $\tilde{\delta}_t := \delta_{(x,t)}$ for all $t \in T$ and then using the earliest available time of a payment as reference point.

Generalized single-switching Changes in preferences are tractable due to a generalized single-switching property. In binary comparisons, a unique threshold captures the postponement beyond which the later payment will be chosen and before which the sooner payment will be chosen. In more general choice sets, this threshold no longer guarantees a choice between the two timed payments but continues to stipulate the point of postponement beyond which the sooner payment cannot be chosen (because the later payment is available) and before which the later payment cannot be chosen (because the sooner payment is available). This generalized single-switching property thus extends our understanding of present bias in binary comparisons to arbitrary choice sets—even in the absence of basic rationality assumptions—and it is closely tied to the unified framework in which references are ordered and preference shifts systematically along this established order.

Present bias as WARP violations Although present bias is typically viewed as a structural violation, ODU predicts a novel manifestation of present bias as WARP violations. Consider the present bias behavior where "\$20 in 4 days" is chosen over "\$18 in 3 days," but "\$18 today" is chosen over "\$20 tomorrow." In ODU, this behavior is explained using a lower discount factor for the latter choice set. However, notice that under this lower discount factor, "\$18 in 3 days" is preferred to "\$20 in 4 days," so the introduction of a third option that induces this discount factor but is not itself chosen, for example "\$15 today," will result in a reversal where "\$18 in 3 days" is chosen over "\$20 in 4 days." This is now a WARP violation that shares the same underlying driver as present bias behavior, even though present bias is typically studied in binary comparisons. In fact, consistent with the spirit of present bias, WARP violations in ODU are restricted to decreased patience, and only when sooner payments are added.

Linking structural properties to basic rationality To further ascertain the aforementioned connection, Proposition 2 shows that relaxing just one of the two conditions would fully recover standard exponential discounting. Consequently, if an ODU DM has *any* utility representation, then she must also have a standard exponential discounting utility representation. This adds to the suggestion that "anomalies" captured by ODU are rooted in systematic changes in preferences. Recall that $\mathcal{D}$ is the collection of all doubletons in $\mathcal{A}$.

Proposition 2. *If c admits an ODU representation, then the following are equivalent:*

[18] Uniqueness is demonstrated in the proof of Theorem 2. When $r = (x, \bar{t})$, δ_r is only used to evaluate alternatives that also arrive at time $\bar{t}$, so any δ_r paired with a strictly increasing u can explain those choices. It could still be unique if $\lim_{t \to \bar{t}} \delta_{(x,t)} = 1$, since an ODU representation requires $\delta_{(x,t)} \leq \delta_{(x,\bar{t})} \leq 1$.

1. *c satisfies WARP over $\mathcal{A}$.*

2. *c satisfies Stationarity over $\mathcal{D}$.*

3. *c admits an exponential discounting utility representation.*

4. *c admits a utility representation.*

Hyperbolic discounting Proposition 2 separates ODU from hyperbolic discounting, quasi-hyperbolic discounting, and related generalizations (Phelps and Pollak, 1968; Loewenstein and Prelec, 1992; Laibson, 1997; Frederick, Loewenstein, and O'donoghue, 2002; Chambers, Echenique, and Miller, 2023; Chakraborty, 2021) due to their adherence to basic rationality, but the empirically informed intuition that discount factors can vary is shared. In contrast, ODU varies discount factors at the choice problem level whereas hyperbolic discounting does so at the alternative level. Binary comparisons hold similar behavioral implications: when two options are gradually advanced, there may be a point where the choice is switched from the sooner to the later.[19] But for larger choice sets, unlike ODU, hyperbolic discounting predicts that the preference ranking between any two options stays the same regardless of the presence of a third alternative.

WARP violations in other time preference settings Beyond the conventional time preference setting, an active literature on menu preference applies Gul and Pesendorfer (2001)'s temptation model to study how smaller menus can act as commitment devices that avoid present bias behavior (Noor, 2011; Lipman, Pesendorfer, et al., 2013; Ahn, Iijima, Le Yaouanq, and Sarver, 2019). In these models, past and future selves prefer to choose differently from the same set of alternatives, which could manifest as a reversal if played out, therefore ODU and these models tackle dynamic inconsistency using related intuitions about long-term and short-term attitudes.

Freeman (2021)'s task completion study, which is related to the above literature and closer to ODU's setting, considers a time-inconsistent DM who exhibits choice reversals when additional opportunities for completions are introduced. In particular, a sophisticated DM ends up completing the task earlier, therefore choosing a sooner option when choice set expands is a common theme between our work. However, the manifestation of this behavior is different; a reversal in ODU can only occur when an alternative earlier than any other is added, yet in Freeman (2021), adding this kind of alternatives either results in the addition chosen or the choice remains unchanged, therefore WARP will hold.

Consumption streams Focusing on one-time payment helps glean the intuition of this framework, but the approach already suggests how an extension to consumption streams, where the DM maximizes $\sum_t \delta^t_{r(A)} u(x_t)$ (for discrete time), can be introduced. If $r(A)$ is the consumption stream that offers the soonest payment, then the characterization amounts to adding Koopmans (1960)'s

[19] Chakraborty (2021) calls this *Weak Present Bias* and studies its implications.

axioms alongside Stationarity (Section 6 discusses adaptable axioms, which include common versions of separability).

5 Ordered Social Utility

Consider a DM whose willingness to share is greater when the situation allows for greater equality. This behavior departs from models of other-regarding preferences that capture a fixed inequality aversion (Fehr and Schmidt, 1999; Bolton and Ockenfels, 2000; Charness and Rabin, 2002). To illustrate, suppose a DM is endowed with $\$10$ and is asked to share it with another individual. However, instead of choosing any split of this $\$10$, she was only given a few options. When asked to choose between giving $\$2$ and giving $\$3$, giving $\$2$ may seem like a fair decision. However, when the choice is between giving $\$2$, $\$3$, or $\$5$, she may opt for giving $\$3$ instead. The choices $c(\{(\$8,\$2),(\$7,\$3)\}) = \{(\$8,\$2)\}$ and $c(\{(\$8,\$2),(\$7,\$3),(\$5,\$5)\}) = \{(\$7,\$3)\}$ violate WARP, and hence a fixed utility function, even if it captures other-regarding preferences and inequality aversion, is incapable of explaining this behavior.

5.1 Preliminaries and axioms

Let $Y = [w,+\infty) \times [w,+\infty)$, where $w \in \mathbb{R}_{>0}$, be a set of income distributions. For each option $(x,y) \in Y$, x is the dollar amount received by the DM and y is the dollar amount given to another individual. Everything else follows Section 2. The first axiom assumes that an income distribution is strictly preferred if it gives someone more and no one less.

Axiom 5.1. *If $x \geq x'$, $y \geq y'$, and $(x,y) \neq (x',y')$, then $c(\{(x,y),(x',y')\}) = \{(x,y)\}$.*

Related in spirit to Axiom 3.2 and Axiom 4.2, the key axiom of this section identifies behavior that conforms with *quasi-linear preferences* when the underlying choice sets have the same level of *attainable equality*.[20]

Definition 8. *c satisfies Quasi-linearity over $\mathcal{S} \subseteq \mathcal{A}$* if for all $A, B \in \mathcal{S}$ and $a \in \mathbb{R}$, if $(x,y) \in c(A)$, $(x',y') \in A$, $(x'+a, y') \in c(B)$, and $(x+a,y) \in B$, then $(x+a,y) \in c(B)$.

The measure of attainable equality is based on the Gini coefficient,

$$G((x,y)) = \frac{|x-y| + |y-x|}{4(x+y)},$$

which ranges from 0 (most balanced) to 0.5 (least balanced) for our 2-agents setting. Analogous to other domains, compliance with Quasi-linearity is called for when choice sets share a Gini-minimizing income distribution.

[20] Since the model involves reference-dependent preferences, using quasi-linear utilities as baseline (rather than using more general models of other-regarding preferences) provides meaningful restrictions.

Definition 9. Let $\Psi(A) := \{(x,y) \in A : G((x,y)) \leq G((x',y')) \text{ for all } (x',y') \in A\}$ be the set of *most-balanced income distributions* in A.

Axiom 5.2 (Equality Reference Dependence)**.** *If $\Psi(A) \cap \Psi(B) \neq \emptyset$, then c satisfies Quasi-linearity over $\{A, B\}$.*

Lemma 2 shows that Axiom 5.2 is a variant of Axiom 3.2, except that Ψ and the behavioral property being generalized have been adapted to the new domain. More generally, it is an application of the meta-framework RD introduced later in Section 6.

Lemma 2. *c satisfies Axiom 5.2 if and only if for every $A \in \mathcal{A}$ and $(x,t) \in \Psi(A)$, c satisfies Quasi-linearity over $\{B \in \mathcal{A} : (x,t) \in B \subseteq A\}$.*

The next and last postulate regulates changes in preferences. Suppose $y > y'$ and a DM chooses to share more (x,y) than to share less (x',y'). I postulate that making more options available will not cause the DM to switch to sharing less, since the added options can only increase attainable equality.

Axiom 5.3. *Suppose $B \subset A$ and $y > y'$. If $(x,y) \in c(B)$ and $(x',y') \in B \backslash c(B)$, then $(x',y') \notin c(A)$.*

5.2 Model

Definition 10. c admits an Ordered Social Utility (OSU) representation if it admits an OU representation $\left(\{U_r\}_{r \in Y}, R\right)$ such that for some strictly increasing functions $\{v_r : [w, +\infty) \to \mathbb{R}\}_{r \in Y}$,

- $U_r(x,y) = x + v_r(y)$,
- $G(r) < G(r')$ implies rRr' and $v_r(y) - v_r(y') \geq v_{r'}(y) - v_{r'}(y')$ for all $y > y'$,
- $G(r) = G(r')$ implies $v_r = v_{r'}$.

Theorem 3. *c satisfies Axioms 5.1-5.3 and Continuity if and only if it admits an OSU representation.*

OSU combines an objective measure of *equality* with a subjective interpretation of *fairness*. Every DM bases her choice on the Gini-minimizing option, $r(A)$, as it captures the amount of attainable equality in a choice set. When attainable equality is higher ($G(r(A))$ is lower), utility difference between sharing more and sharing less increases, reflecting increased willingness to share. The amount of increase depends on the DM's subjective sense of fairness. A very large increase causes WARP violations, where the DM switches from an option that shares less to an option that shares more even though both options are always present. Like in the other domains, preference parameters $\{v_r\}_{r \in Y}$ are unique.[21]

For applications, it is without loss to further simplify OSU by using Gini coefficient—rather than alternatives—to index context-dependent utility from sharing. To do so, for all $\bar{G} \in [0, 0.5)$, set $\tilde{v}_{\bar{G}} := v_{(x,y)}$ where $\bar{G} = G((x,y))$, and then use the lowest attainable *Gini coefficients* as reference points.

[21] Uniqueness is demonstrated in the proof of Theorem 3.

Menu-dependent altruism As in the motivating example, the model explains context-dependent willingness to share when distributing a fixed pie with different splitting options. Suppose a DM is allocating $\$M$ between herself and another individual, and each choice set is characterized by a set of splitting fractions $D \subset [0,1]$. That is, she can allocate $\alpha \cdot \$M$ to herself and $(1-\alpha) \cdot \$M$ to other party if and only if $\alpha \in D$. Consider $D = \{0.6, 0.7\}$ and $D' = \{0.5, 0.6, 0.7\}$. Since attainable equality is greater in D' (it contains an equal split), a DM who chooses 0.7 from D may exhibit increased willingness to share that results in choosing 0.6 from D', even if this violates WARP. But the model rules out the opposite behavior: A DM who chooses 0.6 from D cannot choose 0.7 from D', since it would imply decreased willingness to share. Also, a reversal cannot happen between $D = \{0.6, 0.7\}$ and $D'' = \{0.6, 0.7, 0.8\}$ since they have the same level of attainable equality.

Equality over generosity Willingness to share is maximized when a perfectly balanced income distribution is available. In particular, the model captures increased altruism not due to the opportunity to *give more* per se, but due to the opportunity to *be equal*. To illustrate the difference, consider the same example but with $D = \{0.5, 0.3, 0.2\}$ and $D' = \{0.3, 0.2\}$. Even though D contains alternatives that achieve greater equality, the DM's ability to give is the same across the two choice sets. Yet, since the feasible allocations are always unfavorable to her, higher attainable equality results from her ability to take more. In this example, the DM may be interpreted as being less altruistic when the world is unfair to her, but becomes more altruistic when more greater equality becomes possible.

Fairness over efficiency Consider one last application where OSU allows for willingness to forgo a greater total surplus in favor of sharing. Suppose the DM must choose between $(\$30, \$20)$ and $(\$60, \$0)$. The second option is appealing in that the total amount of money is greater, whereas the first option sacrifices both total surplus and payment to oneself in order to provide a share to the other individual. Suppose $(\$60, \$0)$ is chosen. In OSU, adding $(\$25, \$25)$ as an option can cause the DM to switch from $(\$60, \$0)$ to $(\$30, \$20)$ due to increased altruism. While this behavior seems reasonable, it is inconsistent with any model that complies with WARP.

Empirical evidence The vast literature on distributional preferences provides suggestive evidence for OSU behavior. Moreover, unlike the case of risk and time domains, they do focus on basic rationality violations. In dictator games, List (2007); Bardsley (2008); Korenok, Millner, and Razzolini (2014) find that changes to a dictator's choice set affect her willingness to give and result in WARP-violating choices. Dana, Cain, and Dawes (2006) investigate the underlying mechanism by making the dictator game an option and Dana, Weber, and Kuang (2007) do so by manipulating the visibility of the choice set. They find the audience effect, where fair behavior is the result of subjects' desire to be perceived (by themselves and others) as fair. Rabin (1993) studies an intention-based explanation in game theoretic settings where kindness is reciprocated. Although existing studies motivate OSU, the model does not distinguish between willingness to share that depends intrinsi-

cally on outcomes and that resulting from intentions.[22]

In a more recent study, Cox, List, Price, Sadiraj, and Samek (2016) conduct experiments that explicitly test for basic rationality violations in dictator games and, consistent with OSU, find that shrinking a choice set results in WARP violations in the direction of keeping more for oneself. They propose a modification to basic rationality by introducing a testable prediction based on a definition of moral reference points, which depend on the framing of the problem (e.g., "Give" and "Take") and features of the feasible distributions. When moral reference points are fixed, rationality postulates are satisfied; otherwise, violations favor the party who benefits from the new moral reference point. Their work provides empirical support for OSU, which in turn offers a theory that complements their findings.

Observable contexts and menu preference The intuitions contained in OSU resonates with other studies that, unlike OSU, exploit a richer setting. In settings that include multiple actors, Cox, Friedman, and Sadiraj (2008) study how the generosity of a first mover affects the altruism of a second mover. Cheung (2023) focuses on a second mover who, more generally, makes different decisions from the same choice set based on how the underlying choice set was chosen by a first mover. Relatedly, van Bruggen, Heufer, and Yang (2023) consider a DM whose social preference depends on exogenous contexts like "selfish" and "generous." In a menu preference setting, Dillenberger and Sadowski (2012) study a DM who has shame concern and prefers a smaller menu that excludes normatively better allocations that entail lower self-payoffs, since not choosing those options can induce shame.

Linking structural properties to basic rationality Like in the other domains, Proposition 3 shows that WARP violation and failure of standard postulate (Quasi-linearity) are linked. In this setting, it also suggests that wealth effects may be in part contributed by reference dependent preferences. Recall that $\mathcal{D}$ is the collection of all doubletons in $\mathcal{A}$.[23]

Proposition 3. *If c admits an OSU representation, then the following are equivalent:*

1. *c satisfies WARP over $\mathcal{A}$.*
2. *c satisfies Quasi-linearity over $\mathcal{D}$.*
3. *c admits a quasi-linear utility representation.*
4. *c admits a utility representation.*

[22] More on outcome-based vs intention-based inequality aversion can be found in Ainslie (1992), Nelson (2002), Fehr and Schmidt (2006), Sutter (2007), and Kagel and Roth (2016).

[23] Quasi-linear utility in wealth is often interpreted as the absence of wealth effects. In this domain, it means an individual's willingness to give does not depend on how much she would have left—her wealth—because if giving t is better than giving t' with a base wealth w, i.e., $(w-t)+v(t) > (w-t')+v(t')$, then the same holds true at a different wealth level w', i.e., $(w'-t)+v(t) > (w'-t')+v(t')$.

6 General Framework

The models in all three domains share a "meta" axiomatic framework that in its simplest form characterizes OU. The core of this approach is the generalization of an arbitrarily given behavioral postulate by calling for its local compliance, a general notion called *Reference Dependence* (RD). Given a behavioral postulate of interest $\mathcal{T}$ and an objective range in which reference points lie Ψ, formalized later in this section, RD follows the template "for every choice set A, the choice correspondence c satisfies $\mathcal{T}$ over $\{B \in \mathcal{A} : x \in B \subseteq A\}$ for some $x \in \Psi(A)$."

To illustrate the basic idea, consider the following definition that maintains the behavioral restriction in WARP but allows for its selective application to a subset of problems $\mathcal{S} \subseteq \mathcal{A}$.

Definition 11. *c satisfies WARP over $\mathcal{S} \subseteq \mathcal{A}$* if for all $A, B \in \mathcal{S}$, if $B \subset A$ and $c(A) \cap B \neq \emptyset$, then $c(A) \cap B = c(B)$.

The classical rationality assumption is to impose WARP over $\mathcal{S} = \mathcal{A}$. The following postulate imposes WARP locally using a reference point as the anchor and obtains the OU representation introduced in Section 2.[24]

Axiom 6.1. *For every choice set $A \in \mathcal{A}$, c satisfies WARP over $\{B \in \mathcal{A} : x \in B \subseteq A\}$ for some $x \in A$.*

Theorem 4. *c satisfies Axiom 6.1 and Continuity if and only if it admits an OU representation.*

Axiom 6.1 is first introduced in Rubinstein and Salant (2006) as the *reference point property*. To interpret this axiom, suppose choices from A and its subset B constitute a WARP violation. If this is caused by a change in reference point, specifically, that the reference alternative of A was removed when we go to subset B, then a natural limitation of WARP violations would arise: Had we not removed the reference alternative of A, choices must satisfy WARP. To put it differently, suppose that by preserving some alternative x in A, choices from the subsets of A would comply with WARP, then x is a candidate reference alternative of A. Axiom 6.1 demands that every choice set contains (at least) one candidate reference alternative, which makes it less demanding than the standard postulate that imposes WARP globally.

The remainder of this section introduces RD formally, starting with a general definition of a generic behavioral postulate, and the key result (Lemma 3) that provides the meta-framework for Theorem 1, Theorem 2, Theorem 3, and Theorem 4.

Given Y and $\mathcal{A}$ (Section 2), let $\mathcal{C}$ be the set of all *general choice correspondences* $c : \mathcal{B} \to \mathcal{A}$ such that $\mathcal{B} \subseteq \mathcal{A}$ and $c(B) \subseteq B$ for all $B \in \mathcal{B}$. For a general choice correspondence with domain $\mathcal{A}$, we simply call it a *choice correspondence*. Call $\hat{c} : \mathcal{S} \to \mathcal{A}$ a *subcorrespondence* of $c : \mathcal{B} \to \mathcal{A}$ if $\mathcal{S} \subseteq \mathcal{B}$ and

[24] Kıbrıs, Masatlioglu, and Suleymanov (2023) and Giarlotta, Petralia, and Watson (2023) propose a related generalization of WARP: "if dropping x in the presence of y causes a WARP violation (a reversal), then dropping y in the presence of x cannot." Their formulation is not immediately adaptable to richer domains: Let p', q' be common mixtures of p, q. Using notation $\{\underline{p}, q\}$ to denote "p is chosen from the choice set $\{p, q\}$," the choices $\{\underline{p}, q, p', q'\}$, $\{\underline{p}, q, p'\}$, $\{\underline{p}, q, q'\}$, $\{\underline{p}, p', q'\}$, $\{\underline{q}, p', l'\}$, $\{p, \underline{q}\}$, $\{\underline{p}, p'\}$, $\{\underline{p}, q'\}$, $\{\underline{q}, p'\}$, $\{\underline{q}, q'\}$, $\{p', \underline{q}'\}$ satisfy their axioms even after counting Independence violations as reversals, but there is no reference alternative in $\{p, q, p', q'\}$ because $\{\underline{p}, q, p', q'\}$ and $\{p, \underline{q}\}$ violate WARP whereas $\{\underline{p}, q, p', q'\}$ and $\{p', \underline{q}'\}$ violate Independence. Axiom 3.2 excludes this behavior.

$\hat{c}(B) = c(B)$ whenever defined. If, furthermore, $\mathcal{S}$ is finite, then call $\hat{c}$ a *finite subcorrespondence* of c. A behavioral postulate imposed on general choice correspondences can be identified using a subset $\mathcal{T}$ of $\mathcal{C}$ where some general choice correspondences are admitted and others excluded. In line with how behavioral postulates are typically introduced, I focus on postulates that are easier to satisfy when fewer observations are considered, and call them *theories*.

Definition 12.

1. $\mathcal{T} \subseteq \mathcal{C}$ is a *theory* if for all $c \in \mathcal{T}$, every subcorrespondence of c is in $\mathcal{T}$.
2. $\mathcal{T} \subseteq \mathcal{C}$ is a *finite theory* if it is a theory and for all $c \in \mathcal{C} \backslash \mathcal{T}$, there exists a finite subcorrespondence of c that is not in $\mathcal{T}$.

Postulates that place restrictions on finitely many choice sets at a time are finite theories, such as the common definitions of WARP, monotonicity, transitivity, convexity, betweenness, separability, independence, stationarity, and many others. These are postulates where non-compliance can always be concluded using finitely many observations. Note that an empirically falsifiable property need not be a finite theory, but a finite theory is empirically falsifiable unless it is trivial (i.e., $\mathcal{T} = \mathcal{C}$).[25] Non-examples include various versions of continuity and infinite acyclicity since they require an infinite number of observations to substantiate a violation. When Y is finite, every theory is trivially a finite theory. Imposing multiple postulates, $\mathcal{T}_1$ and $\mathcal{T}_2$, is equivalent to taking the intersection $\mathcal{T}_1 \cap \mathcal{T}_2 \subseteq \mathcal{C}$. Because taking intersection of theories (resp. finite theories) yields a theory (resp. finite theory), this characterization can simultaneously account for multiple postulates (or, a model).[26]

The next result introduces RD, a reference-dependent generalization of a generic behavioral postulate $\mathcal{T}$, and shows that it is equivalent to a representation in which observations are partitioned using a reference order R such that $\mathcal{T}$ holds within each part. Given $c : \mathcal{B} \to \mathcal{A}$, if its subcorrespondence $\hat{c} : \mathcal{S} \to \mathcal{A}$ is in $\mathcal{T}$, I say "c satisfies $\mathcal{T}$ over $\mathcal{S}$." A correspondence $\Psi : \mathcal{B} \to \mathcal{A}$ where $\Psi(A) \subseteq A$ is called an α*−correspondence* if for all $A, B \in \mathcal{B}$, if $a \in \Psi(A)$ and $a \in B \subset A$, then $a \in \Psi(B)$. Given a linear order (R, Y), let $r(A)$ denote the unique element $x \in A$ such that xRy for all $y \in A$. A linear order (R, Y) is called Ψ*-consistent* if for all $A \in \mathcal{B}$, $r(A) \in \Psi(A)$.

Lemma 3. *Consider a choice correspondence c, a finite theory $\mathcal{T}$, and an α−correspondence Ψ. The following are equivalent:*

1. *Reference Dependence (RD): For every choice set $A \in \mathcal{B}$, c satisfies $\mathcal{T}$ over $\{B \in \mathcal{B} : x \in B \subseteq A\}$ for some $x \in \Psi(A)$.*

[25] It is commonly understood that an empirically falsifiable property is one that can be falsified with finitely many observations (i.e., there exists $c \in \mathcal{C} \backslash \mathcal{T}$ such that $|\text{dom}(c)| < \infty$). Consider the combination of WARP and some version of continuity, it is a theory, and it is empirically falsifiable since WARP needs just two observations to falsify. Yet in the absence of WARP violations, a choice correspondence can still violate continuity, which is a non-compliance that cannot be substantiated with finitely many observations.

[26] **Theory**: Consider any $c \in \mathcal{T}_1 \cap \mathcal{T}_2$. For any $\hat{c} \in \mathcal{C}$ where $\hat{c} \subset c$, since $\mathcal{T}_1, \mathcal{T}_2$ are theories, we have $\hat{c} \in \mathcal{T}_1, \mathcal{T}_2$, and hence $\hat{c} \in \mathcal{T}_1 \cap \mathcal{T}_2$, so $\mathcal{T}_1 \cap \mathcal{T}_2$ is a theory. **Finite theory**: Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are finite theories, which are theories, and so $\mathcal{T}_1 \cap \mathcal{T}_2$ is a theory. Consider any $c \in \mathcal{C} \backslash (\mathcal{T}_1 \cap \mathcal{T}_2)$. Without loss of generality say $c \notin \mathcal{T}_1$, so by definition of finite theory we can find a finite subcorrespondence $\hat{c}$ of c where $\hat{c} \notin \mathcal{T}_1$, which means $\hat{c} \notin \mathcal{T}_1 \cap \mathcal{T}_2$.

2. *There exists a Ψ-consistent linear order (R, Y) such that for all $x \in Y$, c satisfies $\mathcal{T}$ over $\{B \in \mathcal{B} : r(B) = x\}$.*

If Ψ is the identity function, then the first condition in Lemma 3 is satisfied when, for each choice problem A, some alternative $x \in A$ serves as an anchor that guarantees compliance with $\mathcal{T}$ in subsets of A. In anticipation, this anchor is a potential reference alternative for A, so the condition can be understood as "there is a reference in every A." When Ψ is not the identity function, we further demand that a potential reference alternative can be found in a predetermined subset of the choice problem, $\Psi(A) \subseteq A$, making reference formation less subjective. The case of fully objective reference is captured when $\Psi(A)$ is a singleton for all A, since it fully pins down the reference. RD is always falsifiable (whenever $\mathcal{T}$ is) and can be written without using an existential quantifier.[27]

7 Related Literature

The earliest literature on context effects considers a DM who does not assign a persistent utility to an alternative but rather evaluates it by explicit comparisons with available alternatives (e.g., attraction, compromise, salience, and focusing) or with a reference point (e.g., loss aversion, status quo bias).[28] Differently, this paper falls into a recent trend that studies context effects even if it does not involve explicit comparisons. The baseline model, if restricted to generic finite alternatives, can be traced back to *triggered rationality* in Rubinstein and Salant (2006).[29] The present paper indispensably generalizes its axiomatic foundation to develop a meta-framework that captures ordered context effects in risk, time, and social preferences. Also taking an umbrella approach using reference dependence, Ellis and Masatlioglu (2022) study *categorical thinkers* who evaluate alternatives after reference-dependent category assignments; however, their work builds on a completely different framework with neither ordered context effects nor canonical functional forms. Closer to the present study, Kovach (2020) studies *wishful thinkers* who evaluate acts using expected utility functionals, thereby preserving canonical functional forms, but with reference-dependent subjective beliefs that deliver context effects.

Ordered context effects are also closely related to a growing literature on random utility and random choice that considers ordered types (Apesteguia, Ballester, and Lu, 2017; Apesteguia and Ballester, 2018; Filiz-Ozbay and Masatlioglu, 2023; Apesteguia and Ballester, 2023; Chambers, Masatlioglu, and Yildiz, 2025), sharing in common the tractability afforded by restricted heterogeneity. Indeed, the notion of order in Apesteguia, Ballester, and Lu (2017) is consistent with the

[27] Falsifiability because each choice set $A \in \mathcal{B}$ is finite. The current formulation with an existential quantifier the current formulation best describes a universal template of *reference-dependent generalization*. Applications of this formulation without an existential quantifier are considered in Section 4 (time preference) and Section 5 (social preference).

[28] See attraction effect (Huber, Payne, and Puto, 1982), compromise effect (Simonson, 1989), salience (Bordalo, Gennaioli, and Shleifer, 2012, 2013), focusing (Kőszegi and Szeidl, 2013)), loss aversion (Kahneman and Tversky, 1979; Kőszegi and Rabin, 2006), and status quo bias (Masatlioglu and Ok, 2005, 2014). See also Tversky and Kahneman (1991); Kahneman, Knetsch, and Thaler (1991); Rubinstein and Zhou (1999); Sagi (2006); Apesteguia and Ballester (2009); Ortoleva (2010); Dean, Kıbrıs, and Masatlioglu (2017); Freeman (2017); Tserenjigmid (2019).

[29] *Triggered rationality* is further investigated by Kıbrıs, Masatlioglu, and Suleymanov (2023); Giarlotta, Petralia, and Watson (2023) with different characterizations and applications.

monotonic changes in risk, time, and social preferences considered in this paper, and the more general notion in Filiz-Ozbay and Masatlioglu (2023) suggests future directions where ordered context effects apply to bounded rationality. A stark difference, however, is that the DM in this paper uses only one, deterministic utility function for every choice problem despite having an ordered set of utility functions. Because choices are deterministic and locally canonical, the framework also differs from existing notions of "multiple selves" (Dubra, Maccheroni, and Ok, 2004; Hara, Ok, and Riella, 2019) and more general forms of random utility (Gul and Pesendorfer, 2006; Lu and Saito, 2018). It is as if the DM has "multiple selves" but only one is activated based on the features of the underlying choice problem, capturing the novel behavior where choices are locally canonical but ordered in aggregate.

Methodologically, this paper introduces a new way to characterize endogenous reference formation that preserves canonical functional forms. A well-known technical challenge is that canonical axioms are insufficient under limited datasets, but ordered reference can impart sufficient richness to each (reference-dependent) collection of behavior to obtain a canonical representation without strengthening axioms.[30] This is conceptually related to the characterization of local expected utility using *weak local independence* (Ke and Zhao, 2024). Building on ordered reference, context effects (e.g., more risk averse, more impatient) are intuitively characterized by simply relating small and large choice sets, thereby capturing the behavioral content of *negative certainty independence* (Dillenberger, 2010; Cerreia-Vioglio, Dillenberger, and Ortoleva, 2015) and *weak present bias* (Chakraborty, 2021) in a more general setting.

More broadly, the paper contributes to a growing interest that focuses on the comprehensive examination of non-standard behaviors, which includes correlated biases (Burks, Carpenter, Goette, and Rustichini, 2009; Dean and Ortoleva, 2019; Chapman, Dean, Ortoleva, Snowberg, and Camerer, 2023; Stango and Zinman, 2023), assessing a broad spectrum of "anomalies" as potential mistakes (Nielsen and Rehbeck, 2022), linking non-standard risk and time preferences (Chakraborty, Halevy, and Saito, 2020), separating preference inconsistency and parametric misspecification (Halevy, 2015; Polisson, Quah, and Renou, 2020; Echenique, Imai, and Saito, 2020, 2023; de Clippel and Rozen, 2023), and revealed preference analyses that focus on basic rationality in rich/different domains (Dembo, Kariv, Polisson, and Quah, 2021; Halevy, Walker-Jones, and Zrill, 2023; Chen, Liu, Shan, Zhong, and Zhou, 2023). From a characterization perspective, the study of choice models may have rightly begun by theoretically decomposing complex behavior into axioms with distinct interpretations—such as "basic rationality" and "structural assumptions"—which may have led to their separate examinations in subsequent work. This paper explores how the two notions may be intuitively related within a framework of ordered context-dependent preferences.

[30] See Samuelson (1948); Aumann (1962) for the insufficiency of canonical axioms and see Houthakker (1950); Echenique, Imai, and Saito (2020); de Clippel and Rozen (2021) for examples of strengthening axioms.

8 Conclusion

This paper presents a single, unifying framework for ordered context-dependent preferences and discusses novel behavior. The key innovation, RD, provides a way to systematically weaken postulates even if they are conceptually distinct. The method is then applied to the risk, time, and social domains. In each setting, behavior can be understood as resulting from canonical models provided that reference points are fixed, and deviations from these models are accounted for by monotonic changes in preference parameters. Reference points in this framework are determined by the maximization of a reference order, which can be viewed as an instrument that captures the relevant context of a choice problem.

The meta-framework can be applied to other domains to capture other intuitive forms of ordered context effects. For example in the case of subjective risk, ambiguity aversion can change from a menu to another as the set of prior expands/shrinks, perhaps because the availability of a (more) constant act causes the DM to be more ambiguity averse.

Building upon a long literature of domain-specific research on seemingly independent structural "anomalies," including but not limited to the Allais paradox and present bias behavior, this paper studies a possible link that relate them to WARP violations. This, in turn, informs more fundamentally on the relationship between rationality postulates and structural postulates. The exercise proposes tractable models for ordered context effects, offers new ways to introduce behavioral postulates, suggests new avenues for empirical research, and adds to our understanding of why normative postulates fail.

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A Appendix: Proofs

Theorems 1, 2, and 3 use Theorem 4 as a prerequisite.

A.1 Proof of Theorem 4

The main intuition of this result can be illustrated using an ad-hoc proof for the case of finite Y.

Lemma 4. *Suppose Y is finite. A choice correspondence c satisfies Axiom 6.1 if and only if it admits an OU representation.*

Proof. "**If**" is straightforward. I prove "**only if.**" Denote by $\Gamma(A)$ the set of alternatives x in A such that c satisfies WARP over $\mathcal{S} = \{B \subseteq A : x \in B\}$, guaranteed to be non-empty by Axiom 6.1. Create a list in the following way: List elements of $\Gamma(Y)$ with an arbitrary order. Since $Y \backslash \Gamma(Y)$ is again finite, continue listing elements of $\Gamma(Y \backslash \Gamma(Y))$ with an arbitrary order; continue until every $x \in Y$ is listed. Finally, let i_x denote the position of x in the list. For any $x, y \in Y$, construct xRy if $i_x \geq i_y$.

For each $x \in Y$, it maximizes R among alternatives in $R^{\downarrow}(x) := \{y : xRy\}$, hence by construction c satisfies WARP over $\mathbb{A}^x_{R^{\downarrow}(x)} = \{A \in \mathcal{A} : r(A) = x\}$. Now construct $(\succsim_x, Y)$. Set $y \succsim_x y$ for all $y \in Y$. For each $y \in R^{\downarrow}(x)$, since $\{x, y\} \in \mathbb{A}^x_{R^{\downarrow}(x)}$, we set $y \succsim_x x$ or $x \succsim_x y$ or both according to $c(\{x, y\})$. For each $y_1, y_2 \in R^{\downarrow}(x)$ such that $y_1 \succsim_x x$ and $y_2 \succsim_x x$, since $\{x, y_1, y_2\} \in \mathbb{A}^x_{R^{\downarrow}(x)}$, we set $y_1 \succsim_x y_2$ or $y_2 \succsim_x y_1$ or both according to $c(\{x, y_1, y_2\})$, this is guaranteed by the fact that c satisfies WARP over $\mathbb{A}^x_{R^{\downarrow}(x)}$. Now, $\succsim_x$ is complete on the set $\mathbb{P}^x := \{y : y \succsim_x x\} \equiv \left\{y \in R^{\downarrow}(x) : y \in c(\{x, y\})\right\}$, which we call the prediction set of x. Now consider $Y \backslash \mathbb{P}^x = \{y : yRx \text{ or } x \succ_x y\}$. Set $y_1 \sim_x y_2$ for all $y_1, y_2 \in Y \backslash \mathbb{P}^x$ and set $y_1 \succ_x y_2$ for all $y_1 \in \mathbb{P}^x$, $y_2 \in Y \backslash \mathbb{P}^x$. The constructed $(\succsim_x, Y)$ is now complete. For transitivity, suppose $y_1 \succsim_x y_2$ and $y_2 \succsim_x y_3$, and that $y_1, y_2, y_3 \in \mathbb{P}^x$ (if any of them is in $Y \backslash \mathbb{P}^x$ then the argument is straightforward by $\sim_x$), hence $y_1 \in c(\{x, y_1, y_2\})$ and $y_2 \in c(\{x, y_2, y_3\})$. Furthermore, since $y_1, y_2, y_3 \in \mathbb{P}^x$, we have $\{x, y_1, y_2, y_3\} \in \mathbb{A}^x_{R^{\downarrow}(x)}$, and c satisfies WARP over $\mathbb{A}^x_{R^{\downarrow}(x)}$ implies $y_1 \in c(\{x, y_1, y_2, y_3\})$, and hence $y_1 \in c(\{x, y_1, y_3\})$, which implies $y_1 \succsim_x y_3$. So $(\succsim_x, Y)$ is transitive.

Finally, we show that (R, Y) and $\{(\succsim_x, Y)\}_{x \in Y}$ explain c. For any $A \in \mathcal{A}$, since A is finite and R is a linear order, there is a unique $R-$maximizer $x \in A$, hence $A \in \mathbb{A}^x_{R^{\downarrow}(x)}$. Suppose for contradiction $y_1 \in c(A)$ but $y_1 \notin \{y \in A : y \succsim_x z \, \forall z \in A\}$, so $y_2 \succ_x y_1$ for some $y_2 \in A$. Then $y_1 \notin c(\{x, y_1, y_2\})$. Since $\{x, y_1, y_2\}$ is a subset of A, and both are in $\mathbb{A}^x_{R^{\downarrow}(x)}$, this is a violation of the statement c satisfies WARP on $\mathbb{A}^x_{R^{\downarrow}(x)}$, hence a contradiction. Suppose for contradiction $y_2 \in \{y \in A : y \succsim_x z \, \forall z \in A\}$ but $y_2 \notin c(A)$. Consider any $y_1 \in c(A)$, since $y_2 \succsim_x y_1$, $y_2 \in c(\{x, y_1, y_2\})$. Since $\{x, y_1, y_2\}$ is a subset of A, and both are in $\mathbb{A}^x_{R^{\downarrow}(x)}$, this is a violation of the statement c satisfies WARP on $\mathbb{A}^x_{R^{\downarrow}(x)}$. Hence $c(A) = \{y \in A : y \succsim_x z \, \forall z \in A\}$. It remains to show that each $\succsim_x$ can be represented by a utility function, but this is standard since Y is finite and $\succsim_x$ is complete and transitive. □

Now we prove the general case where Y is not necessarily finite. "**If**" is straightforward. I prove "**only if.**" Using Lemma 3 (proved in Online Appendix B), let $\mathcal{T}$ be WARP and let Ψ be the

identify function, then there exists a linear order (R, Y) such that c satisfies WARP over $\mathbb{A}^x_{R^{\downarrow}(x)} = \{A \in \mathcal{A} : r(A) = x\}$ for every $x \in Y$. It is obviously Ψ-consistent. Proceed to build $\{(\succsim_x, Y)\}_{x \in Y}$ using the method outlined in the proof of Lemma 4, which gives us a complete and transitive $\succsim_x$ for each x such that $c(A) = \{y \in A : y \succsim_{r(A)} z \, \forall z \in A\}$.

It remains to show that each $\succsim_x$ can be represented by a utility function. Based on our construction, $\succsim_x$ is complete and transitive on Y. Moreover, it is continuous ($y_n \to y$, $z_n \to z$, and $y_n \succsim_x z_n$ for each n implies $y \succsim_x z$) when restricted to the prediction set $\mathbb{P}^x$, otherwise a contradiction of Continuity would be detected in the choices from a sequence of choice problems of form $\{x, y_n, z_n\}$ that converges to $\{x, y, z\}$ ($\mathbb{P}^x$ guarantees that x will not be the only one chosen in any of these sets, so that a contradiction of, say, $z \succ_x y$, will be substantiated in choice: $z \in c(\{x, y, z\})$) . Therefore, along with the fact $\mathbb{P}^x$ is a subset of the separable metric space Y, $\succsim_x$ admits a (continuous) utility function $U : \mathbb{P}^x \to [0, 1]$ that represents $\succsim_x$ when restricted to the alternatives in $\mathbb{P}^x$. Now define $U(z) = -1$ for all $z \in Y \backslash \mathbb{P}^x$. Now U also represents $y \succ_x z$ for all $y \in \mathbb{P}^x$ and $z \in Y \backslash \mathbb{P}^x$ and $z \sim_x z'$ for all $z, z' \in Y \backslash \mathbb{P}^x$. And we are done. Finally, since our system of (R, Y) and $\{(\succsim_x, Y)\}_{x \in Y}$ explains c, which satisfies Continuity, so c has a closed graph.

A.2 Proof Outline of Theorems 1, 2, 3

The proofs for these theorems are repetitive with domain-specific details, I outline the key ideas here and provide complete proofs in Online Appendix B.

Step 1: Reference order R

In their respective domains, Definition 3, Definition 6, and Definition 9 prescribe Ψ's that are α-correspondences. Moreover, WARP, Independence, Stationarity, and Quasi-linearity are finite theories. Therefore, *Risk Reference Dependence* (Axiom 3.2), *Time Reference Dependence* (Axiom 4.2), and *Equality Reference Dependence* (Axiom 5.2) each qualifies as a special case of the "meta" axiom *Reference Dependence*. By invoking Lemma 3, we obtain a linear order (R, Y) that is Ψ-consistent such that for all $r \in \Delta(X)$ (resp. $r \in X \times T$ and $r \in [w, +\infty) \times [w, +\infty)$), c satisfies WARP and Independence (resp. Stationarity, Quasi-linearity) over $\mathbb{A}^r_{R^{\downarrow}(r)} = \{A \in \mathcal{A} : r(A) = x\}$, where the additional compliance with WARP is implied by taking $\alpha = 1$ in the definition of Independence (resp. $a = 0$ in Stationarity and $a = 0$ in Quasi-linearity).

Step 2: Fixed reference, standard representation

Next is to show that for each alternative $r \in Y$, the subcorrespondence $\left(c, \mathbb{A}^r_{R^{\downarrow}(r)}\right)$ admits a standard representation of its respective domain (i.e., expected utility, exponential discounting, quasi-linear utility). This is not obvious; for example in the risk domain, c satisfies WARP and Independence (and Continuity) over $\mathbb{A}^r_{R^{\downarrow}(r)}$, which is a strict subset of all choice problems, so standard

postulates could be insufficient.[31] This issue is resolved by exploiting the structure provided by a Ψ-consistent linear order. In each domain, it guarantees that (for each $r \in Y$) the strict prediction set $\mathbb{P}^r_+ := \left\{p \in R^{\downarrow}(r) : c(\{p,r\}) = \{p\}\right\}$ is rich in a sense that behavior inconsistent with the structural properties can always be substantiated with observations from within $\left(c, \mathbb{A}^r_{R^{\downarrow}(r)}\right)$. For example in the risk domain, we first show that an expected utility representation, with u_r, can be obtained for subcorrespondence $(c, \mathbb{A}^r_{\mathbb{P}})$ where $\mathbb{P}$ is a subset of $\mathbb{P}^r_+$ and is a linear transformation of a $|X|-1$ dimensional simplex set; the existence of $\mathbb{P}$ is given by the Ψ-consistent linear order, which determines which alternatives are in $R^{\downarrow}(r)$ and in turn determines which choice sets are in $\mathbb{A}^r_{R^{\downarrow}(r)}$. Then, for p, q in $R^{\downarrow}(r)$ but possibly outside $\mathbb{P}$, if $\arg\max_{z\in\{p,q\}} \mathbb{E}_z u_r(x) = \{p\}$, it can be shown that there exist common mixtures $p' = p^{\alpha}s$, $q' = q^{\alpha}s$ in $\mathbb{P}$ such that $c(\{r,p',q'\}) = \{p'\}$, and Independence requires $c(\{r,p,q\}) = \{p\}$ (assuming $c(\{r,p,q\}) \neq \{r\}$). Analogous methods, all derived using features of Ψ-consistent linear orders, guarantee the sufficiency of standard postulates in the time and social preference domains.

Step 3: Reference-dependent preferences

Axiom 3.3, Axiom 4.3, and Axiom 5.3 each provides a "direction" for preference change, along the reference order, that must has been satisfied in the constructed representations. Axiom 3.3, Axiom 4.3, and Axiom 5.3 impose restrictions on behavior when a choice set expands, which necessarily imply that a reference point, if it changes, ranks higher in R. If the constructed representations violate the direction of preference change from reference r to r' where rRr', then it can be shown that there exist choice problems $A \in \mathbb{A}^r_{R^{\downarrow}(r)}$ and $B \in \mathbb{A}^{r'}_{R^{\downarrow}(r')}$ such that $B \subset A$ where Axiom 3.3 / Axiom 4.3 / Axiom 5.3 is violated when we compare $c(A)$ and $c(B)$. Like in Step 2, the existence of axiom-violating choice behavior in the underlying subcorrespondences $\left(c, \mathbb{A}^r_{R^{\downarrow}(r)}\right)$ and $\left(c, \mathbb{A}^{r'}_{R^{\downarrow}(r')}\right)$ is guaranteed by Ψ-consistent linear orders. For the time domain, Axiom 4.4 additionally guarantees a persistent consumption utility, so that reference effect can be summarized by changes in discount factors (in general, both discount factor and consumption utility can change).

A.3 Proof of Lemmas 1, 2

The proof of Lemma 1 is presented. The proof of Lemma 2 can be obtained by simply replacing "Stationarity" with "Quasi-linearity."

Only if: Fix $A \in \mathcal{A}$ and $(x,t) \in \Psi(A)$. Consider $\mathcal{S} = \{B \in \mathcal{A} : (x,t) \in B \subseteq A\}$. For any $B_1, B_2 \in \mathcal{S}$, since $(x,t) \in \Psi(B_1) \cap \Psi(B_2)$, c satisfies Stationarity over $\{B_1, B_2\}$. Therefore, c satisfies Stationarity over $\mathcal{S}$ (because Stationarity are restrictions on pairs of choices). **If**: Take any B_1, B_2 such that $\Psi(B_1) \cap \Psi(B_2) \neq \emptyset$. Take $(x,t) \in \Psi(B_1) \cap \Psi(B_2)$. Consider $A = B_1 \cup B_2$. Since B_1 and B_2 are both finite, A is finite, and therefore $A \in \mathcal{A}$. Since $(x,t) \in \Psi(B_1) \cap \Psi(B_2)$, so

[31] For example, if $c(\{p,q\}) = \{p\}$ and $c(\{p',q'\}) = \{q'\}$ where $p = \frac{1}{2}x_1 \oplus \frac{1}{2}x_2$, $q = \frac{3}{4}x_1 \oplus \frac{1}{4}x_3$, $p' = \frac{1}{2}x_2 \oplus \frac{1}{2}x_3$ and $q' = \frac{1}{4}x_1 \oplus \frac{3}{4}x_3$, then c satisfies WARP and Independence over $\{\{p,q\}, \{p',q'\}\}$ but does not admit an expected utility representation (because, even though the lines pq and $p'q'$ are parallel, p, q are not related to p', q' by a common mixture).

$(x,t) \in \Psi(A)$, and so c satisfies Stationarity over $\{B \subseteq A : (x,t) \in B\}$, which contains B_1 and B_2 by construction and we are done.

A.4 Proof of Propositions 1, 2, 3

I focus on showing that WARP (1) and structural postulate (2) are independently sufficient for the standard model (3). The remaining statements, that WARP and structural postulates are necessary for standard models ((1) if (3) and (2) if (3)), and that WARP is sufficient and necessary for a (general) utility representation ((1) if and only if (4)), are well-known and omitted.

Proof of Proposition 1: (1) / (2) implies (3)

Suppose a choice correspondence c admits an OEU representation with specification $(R, \{u_r\}_r)$. Suppose c satisfies WARP or Independence (or both). We first show that $u_r = u_s$ for all $r, s \in \Delta(X) \setminus \text{conv}(\{\delta_b, \delta_w\})$. Suppose without loss of generality rRs. Suppose for contradiction $u_r \neq u_s$. Before we proceed, observe that if iRj and $u_i \neq u_j$, then the facts that u_i is a concave transformation of u_j and that both functions are normalized to $u_i(w) = u_j(w) = 0$, $u_i(b) = u_j(b) = 1$ imply

$$u_r(x) > u_s(x) \, \forall x \in X \setminus \{b, w\}. \tag{A.1}$$

For any $i \in \Delta(X)$, let $\tau_i := \text{conv}(\{i, \delta_b, \delta_w\})$. Denote by $\text{Int}\tau_i$ the interior of τ_i.

WARP violation: By Axiom 3.1, $c(\{\delta_b, r\}) = c(\{\delta_b, s\}) = \delta_b$. Then by Continuity, there exist open balls around δ_b, B_r and B_s, such that they contain lotteries that are chosen over r and s respectively. Consider an open subset S of $B_r \cap B_s \cap \text{Int}\tau_s$. Note that $\text{Int}\tau_s$ only contains extreme spreads of s, and hence $S \subseteq \text{Int}\tau_s \subseteq R^{\downarrow}(s) \subseteq R^{\downarrow}(r)$. Due to Equation A.1, we can find lotteries $p, q \in S$ such that $\mathbb{E}_p u_r(x) > \mathbb{E}_q u_r(x)$ and $\mathbb{E}_p u_s(x) < \mathbb{E}_q u_s(x)$ (for example, we can let $p := \alpha p^* \oplus (1-\alpha) z$ and $q := \alpha q^* \oplus (1-\alpha) z$ where $z \in S$, p^* only assigns positive probabilities to $x \in X \setminus \{b, w\}$, $q^* = 0.5b \oplus 0.5w$, $\mathbb{E}_{p^*} u_r(x) > 1/2 > \mathbb{E}_{s^*} u_r(x)$ and α is sufficiently small so that $p, q \in S$). Taken together we have

$$c(\{r, s, p, q\}) = \{p\} \text{ and} \tag{A.2}$$

$$c(\{s, p, q\}) = \{q\}, \tag{A.3}$$

which jointly violate WARP (1).

Independence violation: Consider $r' \in \text{conv}(\{\delta_b, \delta_w\})$ that satisfies $\mathbb{E}_{r'} u_r(x) = \mathbb{E}_r u_r(x)$, note that $r' \in \text{Int}\tau_r \subseteq R^{\downarrow}(r)$, hence

$$c(\{r, r'\}) = \{r, r'\}. \tag{A.4}$$

It is established in the proof of the representation theorem that $R^{\downarrow}(r)$ contains a full dimensional convex subset of $\Delta(X)$ (specifically, Lemma 7 of Online Appendix B). Fix one such set, B. Take any $z \in B$ and pick α small enough such that $p := \alpha r \oplus (1-\alpha) z$ and $p' := \alpha r' \oplus (1-\alpha) z$ are both in B. Due to Equation A.1, $\mathbb{E}_p u_s(x) - \mathbb{E}_{p'} u_s(x) > \mathbb{E}_p u_r(x) - \mathbb{E}_{p'} u_r(x) (= 0)$. Since $p, p' \in R^{\downarrow}(s)$,

u_p and $u_{p'}$ are concave transformations of u_s, so $\mathbb{E}_p u_k(x) - \mathbb{E}_{p'} u_k(x) \geq \mathbb{E}_p u_s(x) - \mathbb{E}_{p'} u_s(x)$ for $k \in \{p, p'\}$ (also using Equation A.1 if $u_k \neq u_s$ and is otherwise trivial), hence

$$c\left(\left\{p, p'\right\}\right) = \{p\}. \tag{A.5}$$

Since (p, p') are common mixtures of (r, r'), Equation A.4 and Equation A.5 jointly violate Independence (2).

Next we turn to $r \in \text{conv}(\{\delta_b, \delta_w\})$ and show that u_r is either identical, or has the freedom to be identical, to u_s where $s \in \Delta(X) \setminus \text{conv}(\{\delta_b, \delta_w\})$. If $r = \delta_b$ or $r = \delta_w$ or $R^{\downarrow}(r) \subseteq \text{conv}(\{\delta_b, \delta_w\})$, then any strictly increasing u_r can explain c over $\mathbb{A}^r_{R^{\downarrow}(r)} := \left\{A \in \mathcal{A} : A \subseteq R^{\downarrow}(r) \text{ and } r \in A\right\}$, so we can just pick one that is identical to u_s for every $s \in \Delta(X) \setminus \text{conv}(\{\delta_b, \delta_w\})$. If r doesn't satisfy any of those conditions, then there exists $s_2 \in \Delta(X) \setminus \text{conv}(\{\delta_b, \delta_w\})$ such that r is an extreme spread of s_2 and there exists $s_1 \in R^{\downarrow}(r) \setminus \text{conv}(\{\delta_b, \delta_w\})$. This implies u_{s_2} is a concave transformation of u_r (because $s_2 R r$) and u_r is a concave transformation of u_{s_1} (because $r R s_1$), but we already showed that $u_{s_1} = u_{s_2}$ (since $s_1, s_2 \in \Delta(X) \setminus \text{conv}(\{\delta_b, \delta_w\})$), so u_r is identical to u_s for all $s \in \Delta(X) \setminus \text{conv}(\{\delta_b, \delta_w\})$.

We conclude that if either WARP or Independence (or both) holds, then c admits an expected utility representation.

Proof of Proposition 2: (1) / (2) implies (3)

Suppose a choice correspondence c admits an ODU representation with specification $(\{\delta_r\}_r, u)$. We show that if $\delta_r \neq \delta_{r'}$ for some $r, r' \in [0, \bar{t})$, then c violates both WARP and Stationarity. ($\delta_{\bar{t}}$ only plays a role for choice problems $A \in \mathcal{A}$ where every alternative has $t = \bar{t}$, so we set it as $\delta_{\bar{t}} = \delta_0$.)

Suppose for contradiction $\delta_r \neq \delta_{r'}$ for some $r, r' \in [0, \bar{t})$. Say without loss of generality $r > r' \geq 0$, then $\delta_r > \delta_{r'} \geq \delta_0$. Recall that $X = [a, b]$. Consider alternatives $(b - \Delta_x, 0), (b, 0 + \Delta_t) \in X \times T$ such that (i) $\Delta_x \in (0, b - a)$, (ii) $\Delta_t \in (0, \bar{t} - r)$, (iii) $\delta_r^0 u(b - \Delta_x) < \delta_r^{0+\Delta_t} u(b)$, and (iv) $\delta_0^0 u(b - \Delta_x) > \delta_0^{0+\Delta_t} u(b)$, which is possible due in part to the assumption that $(b, \bar{t}) \in c(\{(a, 0), (b, \bar{t})\})$. Note that (i) and (ii) guarantee $(b - \Delta_x, 0), (b, 0 + \Delta_t), (b - \Delta_x, r), (b, r + \Delta_t) \in X \times T$. Then, (iii) gives

$$c(\{(b - \Delta_x, r), (b, r + \Delta_t)\}) = \{(b, r + \Delta_t)\}, \tag{A.6}$$

and (iv) gives

$$c(\{(b - \Delta_x, 0), (b, 0 + \Delta_t)\}) = \{(b - \Delta_x, 0)\} \text{ and} \tag{A.7}$$

$$c(\{(a, 0), (b - \Delta_x, r), (b, r + \Delta_t)\}) = \{(b - \Delta_x, r)\}, \tag{A.8}$$

where Equation A.8 is due in part to the assumption that $(b, \bar{t}) \in c(\{(a, 0), (b, \bar{t})\})$ as it excludes $(a, 0)$ from being uniquely chosen. Now note that Equation A.6 and Equation A.7 jointly violate Stationarity (2), whereas Equation A.6 and Equation A.8 jointly violate WARP (1). We conclude

that if either WARP or Stationarity (or both) holds, then c admits an exponential discounting utility representation.

Proof of Proposition 3: (1) / (2) implies (3) Suppose a choice correspondence c admits an OSU representation with specification $\{v_r\}_r$. We show that if $v_r(y) - v_r(y') \neq v_{r'}(y) - v_{r'}(y')$ for some r, r' and $y > y'$, then c violates both WARP and Quasi-linearity.

Without loss of generality say $r > r' \geq 0$. Then $v_r(y) - v_r(y') < v_{r'}(y) - v_{r'}(y') \leq v_0(y) - v_0(y')$, and therefore there exist $\tilde{x}' \in [w, +\infty)$ such that (i) $\tilde{x}' + v_r(y') > w + v_r(y)$ and (ii) $\tilde{x}' + v_0(y') < w + v_0(y)$.

Since for any fixed $\bar{y}$, $G((\cdot, \bar{y}))$ is continuous and asymptotically increasing in it's first argument, there exists $\Delta, \Delta^* > 0$ such that $\min(\{G((w+\Delta, y)), G((\tilde{x}'+\Delta, y'))\}) = 0$ and $\min(\{G((w+\Delta^*, y)), G((\tilde{x}'+\Delta^*, y'))\}) = r$. Fix such Δ, Δ^* and let $x := w + \Delta^*$, $x' := \tilde{x}' + \Delta^*$. Due to (i), we have

$$c(\{(x, y), (x', y')\}) = \{(x', y')\}. \tag{A.9}$$

And note that $x' > x \geq w$ and $y > y' \geq w$, so with (ii) we have

$$c(\{(w, w), (x, y), (x', y')\}) = \{(x, y)\}. \tag{A.10}$$

Note that Equation A.9 and Equation A.10 jointly violate WARP (1). Meanwhile, also due to (ii), we have

$$c(\{(x - \Delta^* + \Delta, y), (x' - \Delta^* + \Delta, y')\}) = \{(x - \Delta^* + \Delta, y)\}. \tag{A.11}$$

Note that Equation A.9 and Equation A.11 jointly violate Quasi-linearity (2). We conclude that if either WARP or Quasi-linearity (or both) holds, then c admits a quasi-linear utility representation.

Supplemental Materials

(Online)

B Online Appendix: Omitted Proofs and Results

B.1 Proof of Lemma 3

The proof for (2) implies (1) is straightforward: For every (finite) set $A \in \mathcal{A}$, the maximizer of the linear order R is an "x" in (1). We focus on the proof for (1) implies (2). The proof for (2) implies (1) begins with an observation using Zermelo's well-ordering theorem and transfinite recursion, and then uses it build a reference order given an arbitrary finite theory $\mathcal{T}$ (Definition 12).

Lemma 5. *Let Z be a set and let $\mathbb{Z}$ be the set of all finite and nonempty subsets of Z. Let $\mathcal{R}$ be a self-map on $\mathbb{Z}$ such that $\mathcal{R}(S) \subseteq S$. Suppose for all $T, S \in \mathbb{Z}$ and $x \in Z$ such that $x \in T \subseteq S$, if $x \in \mathcal{R}(S)$, then $x \in \mathcal{R}(T)$ (property α). Then, there exists a self-map $\mathcal{R}^*$ on $\mathbb{Z}$ such that*

(i) For all $S \in \mathbb{Z}$, $\mathcal{R}^*(S) \subseteq \mathcal{R}(S)$.

(ii) For all $T, S \in \mathbb{Z}$ and $x \in Z$ such that $x \in T \subseteq S$, if $x \in \mathcal{R}^*(S)$, then $x \in \mathcal{R}^*(T)$ (property α), and

(iii) For all $S \in \mathbb{Z}$, $|\mathcal{R}^*(S)| = 1$

Proof. We prove this by construction. Assume and invoke Zermelo's theorem (also known as the well-ordering theorem) to well-order the set of all doubletons in the domain of $\mathcal{R}$. Now we start the transfinite recursion using this order.

In the zero case, we have $\mathcal{R}_0 = \mathcal{R}$. This correspondence satisfies α and is nonempty-valued ($\mathcal{R}_0(S) \neq \emptyset$ for all $S \in \mathbb{Z}$).

For the successor ordinal $\sigma + 1$, having supposed $\mathcal{R}_\sigma$ satisfies α and is nonempty-valued, we take the corresponding doubleton $B_{\sigma+1}$ and take $x \in B_{\sigma+1}$ such that $\forall S \supset B_{\sigma+1}$, $\mathcal{R}(S) \setminus \{x\} \neq \emptyset$. Suppose such an x does not exist, then for both $x, y \in B_{\sigma+1}$, there are $S_x \supset B_{\sigma+1}$ and $S_y \supset B_{\sigma+1}$ such that $\mathcal{R}_\sigma(S_x) = \{x\}$ and $\mathcal{R}_\sigma(S_y) = \{y\}$ since $\mathcal{R}_\sigma$ is nonempty-valued. Consider $S_x \cup S_y \in \mathbb{Z}$. Since $\mathcal{R}_\sigma$ is nonempty-valued, $\mathcal{R}_\sigma(S_x \cup S_y) \neq \emptyset$. But since $\mathcal{R}_\sigma$ satisfies α, it must be that $\mathcal{R}_\sigma(S_x \cup S_y) \subseteq \mathcal{R}_\sigma(S_x) \cup \mathcal{R}_\sigma(S_y)$, hence $\mathcal{R}_\sigma(S_x \cup S_y) \subseteq \{x, y\}$. Suppose without loss $x \in \mathcal{R}_\sigma(S_x \cup S_y)$, then due to α again and that $x \in B_{\sigma+1} \subset S_y$, it must be that $x \in \mathcal{R}_\sigma(S_y)$, which contradicts $\mathcal{R}_\sigma(S_y) = \{y\}$. (That is, we showed that with nonempty-valuedness and α, no two elements can each have a unique appearance in the $\mathcal{R}_{(\cdot)}$-image of a set containing those two elements.) Hence, $\exists x \in B_{\sigma+1}$ such that $\forall S \supset B_{\sigma+1}$, $\mathcal{R}(S) \setminus \{x\} \neq \emptyset$. Define $\mathcal{R}_{\sigma+1}$ from $\mathcal{R}_\sigma$ in the following way: $\forall S \supset B_{\sigma+1}$, $\mathcal{R}_{\sigma+1}(S) := \mathcal{R}_\sigma(S) \setminus \{x\}$. Note: (i) Since x is deleted from $\mathcal{R}_\sigma(T)$ only if it is also deleted (if it is in it at all) from $\mathcal{R}_\sigma(S)$ $\forall S \supset T$, we are preserving α, and (ii) since x is never the unique element in $\mathcal{R}_\sigma(S)$ $\forall S \supset B_{\sigma+1}$, we preserve nonempty-valuedness.

For a limit ordinal λ, define $\mathcal{R}_\lambda = \cap_{\sigma<\lambda}\mathcal{R}_\sigma$. Note that since $\mathcal{R}_{\sigma'} \subset \mathcal{R}_{\sigma''}$ $\forall \sigma' > \sigma''$, $\cap_{\sigma \leq \bar{\sigma}} = \mathcal{R}_{\bar{\sigma}}$. Furthermore, for any $\sigma < \lambda$, $\mathcal{R}_\sigma$ is constructed such that α and nonempty-valuedness are preserved. Hence $\mathcal{R}_\lambda$ satisfies α and is nonempty-valued.

Note that this process terminates when all the doubletons have been visited, for we would otherwise have constructed an injection from the class of all ordinals to the set of all doubletons in $\mathbb{Z}$, which is impossible.

Finally, we check that $|\mathcal{R}_\lambda(S)| = 1$ for all $S \in \mathbb{Z}$. Suppose not, hence $\exists S \in \mathbb{Z}$ such that $\{x, y\} \subseteq \mathcal{R}_\lambda(S)$. Then by α we have $\{x, y\} = \mathcal{R}_\lambda(\{x, y\})$, which is not possible as the recursion process has visited $\{x, y\}$ and deleted something from $\mathcal{R}(\{x, y\})$. Now set $\mathcal{R}_\lambda = \mathcal{R}^*$ and we are done. □

For notational convenience, subcorrespondence $\hat{c} : \mathcal{S} \to \mathcal{A}$ of $c : \mathcal{B} \to \mathcal{A}$ is referred to as $(c, \mathcal{S})$, as in "c restricted to $\mathcal{S}$." Given $\mathcal{B} \subseteq \mathcal{A}$, for any $S \subseteq Y$ and $x \in S$, define

$$\mathbb{A}_S^x := \{A \in \mathcal{B} : x \in A \subseteq S\}.$$

Given $\mathcal{T} \subseteq \mathcal{C}$ and a general choice correspondence $c : \mathcal{B} \to \mathcal{A}$, let $\Gamma(S) := \{x \in S : (c, \mathbb{A}_S^x) \in \mathcal{T}\}$ denote the set of *reference alternatives of* S (note that S need not be in $\mathcal{B}$). The following observations are obtained when $\mathcal{T}$ is a finite theory.

Lemma 6. *Let $c : \mathcal{B} \to \mathcal{A}$ be a general choice correspondence and $\mathcal{T}$ a finite theory. Consider $A, B, D \subseteq Y$.*

1. *If $x \in \Gamma(A)$ and $B \subset A$, then $x \in \Gamma(B)$.*
2. *If $x \in \Gamma(A)$ for all finite $A \subseteq D$, then $x \in \Gamma(D)$.*

Proof. Since $B \subseteq A$ implies $\mathbb{A}_B^x \subseteq \mathbb{A}_A^x$ and since $(c, \mathbb{A}_A^x) \in \mathcal{T}$, $(c, \mathbb{A}_B^x) \in \mathcal{T}$ is a direct consequence of the definition of a theory. For (2), suppose for contradiction $x \notin \Gamma(D)$. Because $\mathcal{T}$ is a finite theory, we can find a finite set of choice problems $\mathcal{S} = \{A_1, ..., A_n\} \subseteq \mathbb{A}_D^x$ such that $(c, \mathcal{S}) \notin \mathcal{T}$. Since the set $A := \cup_{i=1}^n A_i \subseteq D$ is finite, $x \in \Gamma(A)$. Note that $\mathcal{S} \subseteq \mathbb{A}_A^x$, so the definition of a theory gives $(c, \mathcal{S}) \in \mathcal{T}$, a contradiction. □

Now I prove (1) implies (2) in Lemma 3. Let $\mathcal{R}' : \mathcal{A} \to \mathcal{A} \cup \{\emptyset\}$ be a set-valued function that picks out reference alternatives, formally $\mathcal{R}'(A) := \{x \in A : (c, \mathbb{A}_A^x) \in \mathcal{T}\}$. Since $\mathcal{T}$ is a finite theory, by point 1 of Lemma 6, $\mathcal{R}'$ satisfies property α (defined in Lemma 5). Furthermore, (1) in Lemma 3 guarantees that $\mathcal{R}'(A) \cap \Psi(A)$ is nonempty for all $A \in \mathcal{A}$. Finally, define $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ by $\mathcal{R}(A) := \mathcal{R}'(A) \cap \Psi(A)$. Since both $\mathcal{R}'(A)$ and $\Psi(A)$ satisfy property α, $\mathcal{R}(A)$ satisfies property α.

Putting the $\mathcal{R}$ we just built through Lemma 5, we get a function $\mathcal{R}^*$ that picks one thing from every set and satisfies property α. With this, we build the order (R, Y) by setting xRy if $\{x\} = \mathcal{R}^*(\{x, y\})$ and xRx for all $x \in Y$. It is well-known that this results in a linear order (R, Y) such that $\mathcal{R}^*(A) = \{x \in A : xRy \, \forall y \in A\}$ for all $A \in \mathcal{A}$. Since $\mathcal{R}^*(A) \subseteq \mathcal{R}(A) \subseteq \Psi(A)$ for all $A \in \mathcal{A}$, this means (R, Y) is also Ψ-consistent.

Finally, consider the set of alternatives that are "reference dominated" by x according to R (including x itself), denoted by

$$R^{\downarrow}(x) := \{y \in Y : xRy\}.$$

For any finite subset $A \subseteq R^{\downarrow}(x)$ such that $x \in A$, we have $x \in \mathcal{R}^*(A) \subseteq \mathcal{R}(A) \subseteq \mathcal{R}'(A)$, which by definition implies x is a reference alternative of A. Using point 2 of Lemma 6, we conclude that x is reference alternative for $R^{\downarrow}(x)$, which need not be finite.

To summarize, we have created a partition of $\mathcal{A}$ where the parts are characterized by $\left\{\mathbb{A}^x_{R^{\downarrow}(x)}\right\}_{x \in Y}$. To see this, take any $A \in \mathcal{A}$, since R is a linear order, there is a unique $z \in A$ such that zRy for all $y \in A$, and so $A \in \mathbb{A}^z_{R^{\downarrow}(z)}$ and $A \notin \mathbb{A}^y_{R^{\downarrow}(y)}$ for any $y \neq z$. Furthermore for each part $\mathbb{A}^x_{R^{\downarrow}(x)}$, $\left(c, \mathbb{A}^x_{R^{\downarrow}(x)}\right)$ is in $\mathcal{T}$. Since $\{B \in \mathcal{A} : r(B) = z\}$ is simply $\mathbb{A}^z_{R^{\downarrow}(z)}$, the proof is complete.

B.2 Proof of Theorem 1

"**If**" is straightforward. I prove "**only if.**" We interpret $\Delta(X)$ as a $|X|-1$ dimensional simplex, and full-dimensional means $|X|-1$ dimensional. Also, where $\text{conv}(\{\delta_b, \delta_w\})$ denotes the set of lotteries that only put non-zero probabilities on prizes b and w, we partition $\Delta(X)$ into three parts: $I = \Delta(X) \setminus \text{conv}(\{\delta_b, \delta_w\})$, $E_1 = \left\{r \in \text{conv}(\{\delta_b, \delta_w\}) : c(\{r, p\}) = \{p\} \text{ for some } p \in R^{\downarrow}(r) \cap I\right\}$, and $E_2 = \text{conv}(\{\delta_b, \delta_w\}) \setminus E_1$. **Stage 1** builds the reference order R. **Stage 2** provides basic results about the prediction set of each reference lottery. **Stage 3** builds a (Bernoulli) utility function for each $r \in I \cup E_1$ and **Stage 4** shows that they are related by concave transformations. **Stage 5** deals with $r \in E_2$.

For any $r \in \Delta(X)$, let $\mathbb{P}^r_+ := \left\{p \in R^{\downarrow}(r) \setminus \{r\} : c(\{p, r\}) = \{p\}\right\}$. For any $r, p \in \Delta(X)$, let $\mathbb{P}^r_{+p} := \left\{q \in R^{\downarrow}(r) \setminus \{r, p\} : c(\{r, p, q\}) = \{q\}\right\}$. We call these *prediction sets*. Note that if rRp, then the fact that c satisfies WARP over $\mathbb{A}^r_{R^{\downarrow}(r)}$ implies $\mathbb{P}^r_{+p} \subseteq \mathbb{P}^r_+$. For any $\mathbb{P} \subseteq \Delta(X)$ and lotteries $p, q \in \Delta(X)$, we call (p', q') a $\mathbb{P}$-*common mixture* of (p, q) if for some $s \in \Delta(X)$ and $\alpha \in [0, 1]$, we have $p' = p^\alpha s$, $q' = q^\alpha s$, and $p', q' \in \mathbb{P}$.

Stage 1: Reference order R

A binary relation R is said to be *risk-consistent* if qRp whenever $p\text{MPS}q$ or $p\text{ES}q$. Note that Ψ is an $\alpha-$correspondence. By Lemma 3, Axiom 3.2 gives a linear order $(R, \Delta(X))$ where c satisfies WARP and Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$ for any $r \in \Delta(X)$ (the additional compliance with WARP is implied by taking $\alpha = 1$ in the definition of Independence). Since R is Ψ-consistent (i.e., $\max(A, R) \in \Psi(A)$) and $\Psi(\{p, q\}) = \{q\}$ if $p\text{MPS}q$ or $p\text{ES}q$, so R is risk-consistent.

Stage 2: Technical Preparations

The next results guarantee that the revealed preference relation constructed using subcorrespondence $\left(c, \mathbb{A}^r_{R^{\downarrow}(r)}\right)$, where r is given, is complete and transitive on a full-dimensional convex subset

of $\Delta(X)$. This is due in large part to R being risk-consistent, and because of it, choices that further satisfy Independence will have an expected utility representation.

Lemma 7. *For any $r \in I$ and any open ball B_r that contains r, $B_r \cap R^{\downarrow}(r)$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. Take any $r \in I$. By definition, $r(x) \neq 0$ for some $x \in X \setminus \{b, w\}$. Consider the set $\mathbb{C}(r) := \{r\} \cup ES(\{r\}) \cup MPS(\{r\} \cup ES(\{r\}))$. It consists of r, all extreme spreads of r, and all of their mean-preserving spreads.

To see $\mathbb{C}(r)$ is convex: First note that since $ES(\{r\})$ is a convex set and r is on the boundary of $ES(\{r\})$, so $\{r\} \cup ES(\{r\})$ is convex. Take any two lotteries $p_1, p_2 \in \mathbb{C}(r)$ and consider their convex combination $(p_1)^{\alpha}(p_2)$ for some $\alpha \in (0,1)$. Since $p_1, p_2 \in \mathbb{C}(r)$, there exist $e_1, e_2 \in \{r\} \cup ES(\{r\})$ such that either $p_1 = e_1$ or $p_1 \text{MPS} e_1$ and either $p_2 = e_2$ or $p_2 \text{MPS} e_2$. If $p_i = e_i$ for both $i = 1, 2$, then $(p_1)^{\alpha}(p_2) = (e_1)^{\alpha}(e_2)$ and by convexity of $\{r\} \cup ES(\{r\})$ we are done. Suppose $p_i \neq e_i$ for some $i = 1, 2$, then since the mean-preserving spread relation is preserved under convex combinations, we have $(p_1)^{\alpha}(p_2) \,\text{MPS}\, (e_1)^{\alpha}(e_2)$. Then, since $(e_1)^{\alpha}(e_2) \in \{r\} \cup ES(\{r\})$ by the convexity of $\{r\} \cup ES(\{r\})$, we have $(p_1)^{\alpha}(p_2) \in MPS(\{r\} \cup ES(\{r\})) \subseteq \mathbb{C}(r)$.

To see $\mathbb{C}(r)$ is full-dimensional: For any $p \in I$, $MPS(\{p\})$ is $|X-2|$ dimensional, and it is a subset of the $|X-2|$ dimensional space defined by lotteries that have the same mean as p. But $ES(\{p\})$ contains lotteries that do not have the same mean as p, and therefore $ES(\{p\}) \cup MPS(\{p\})$ is full-dimensional. This means $\mathbb{C}(r)$ is full dimensional as well since it contains $ES(\{p\}) \cup MPS(\{p\})$ for some $p \in I$.

To see $\mathbb{C}(r) \subseteq R^{\downarrow}(r)$: If $p \in ES(\{r\})$, rRp since R is risk-consistent. If $q \in MPS(\{r\} \cup ES(\{r\}))$, qRp for some $p \in \{r\} \cup ES(\{r\})$ since R is risk-consistent, and by transitivity of R we have qRr. Since B_r is also a full-dimensional and convex set, $B_r \cap \mathbb{C}(r)$ is a full-dimensional convex subset of $B_r \cap R^{\downarrow}(r)$. □

Lemma 8. *For any $r \in I$, $\mathbb{P}^r_+$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. Fix $r \in I$. Note that $\mathbb{P}^r_+$ contains an extreme spread e of r (else, there is a sequence of alternatives $e_k = (\delta_w)^{\alpha_k}(\delta_b)$ such that α_k converges from above to $r(w)$ such that $r \in c(\{r, e_k\})$ for all k, which by Continuity means $r \in c\left(\left\{r, (\delta_w)^{r(w)}(\delta_b)\right\}\right)$, a violation of FOSD (Axiom 3.1)). Consider $q = r^{0.5}e \in I$. Since $q \text{ES} r$, $q \in R^{\downarrow}(r)$. Since c satisfies Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$ and $c(\{r, e\}) = \{e\}$, we establish $q \in \mathbb{P}^r_+ \cap I$. By Continuity, there exists an open ball B_q around q such that $c(\{r, q'\}) = \{q'\}$ for all $q' \in B_q$. By Lemma 7, $B_q \cap R^{\downarrow}(q)$ contains a full-dimensional convex subset of $\Delta(X)$. Moreover, $B_q \cap R^{\downarrow}(q) \subseteq B_q \cap R^{\downarrow}(r) \subseteq \mathbb{P}^r_+$, hence $\mathbb{P}^r_+$ contains a full-dimensional convex subset of $\Delta(X)$. □

Lemma 9. *Fix any $r \in \Delta(X)$. If $p \in \mathbb{P}^r_+ \cap I$, then $\mathbb{P}^r_{+p}$ contains a full-dimensional convex subset of $\Delta(X)$. If $\mathbb{P}^r_+ \cap I \neq \emptyset$, then $\mathbb{P}^r_+$ contains a full-dimensional convex subset of $\Delta(X)$.*

Proof. Fix $r \in \Delta(X)$ and $p \in \mathbb{P}^r_+ \cap I$. Since $p \in I$, the set of extreme spreads of p, $ES(p)$, is nonempty. Also, $ES(p) \subseteq R^{\downarrow}(p) \subseteq R^{\downarrow}(r)$. Since c satisfies WARP over $\mathbb{A}^r_{R^{\downarrow}(r)}$ and $p \in \mathbb{P}^r_+$, $r \notin$

$c(\{r,p,s\})$ for all $s \in R^{\downarrow}(r)$, so we can use the same technique in the proof of Lemma 8 to establish that $\mathbb{P}^r_{+p}$ contains an extreme spread e of p (else, $p \in c\left(\left\{r,p,(\delta_w)^{p(w)}(\delta_b)\right\}\right)$ by Continuity, which violates FOSD (Axiom 3.1)). Consider $q = p^{0.5}e \in I$. Because qESp and $p \in R^{\downarrow}(r)$, so $q \in R^{\downarrow}(r)$. Since c satisfies Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$ and $c(\{r,p,e\}) = \{e\}$, we establish $q \in \mathbb{P}^r_{+p} \cap I$. By Continuity, there exists an open ball B_q around q such that $c(\{r,p,q'\}) = \{q'\}$ for all $q' \in B_p$. By Lemma 7, $B_q \cap R^{\downarrow}(q)$ contains a full-dimensional convex subset of $\Delta(X)$. Moreover, $B_q \cap R^{\downarrow}(q) \subseteq B_q \cap R^{\downarrow}(r) \subseteq \mathbb{P}^r_{+p}$, hence $\mathbb{P}^r_{+p}$ contains a full-dimensional convex subset of $\Delta(X)$. The second statement is given by the first statement and the observation that c satisfies WARP over $\mathbb{A}^r_{R^{\downarrow}(r)}$ implies $\mathbb{P}^r_{+p} \subseteq \mathbb{P}^r_{+}$. □

Stage 3: Expected utility when $r \in I \cup E_1$

Lemma 8 and Lemma 9 establish that when $r \in I \cup E_1$, $\mathbb{P}^r_{+}$ contains a full-dimensional convex subset of $\Delta(X)$. The next result shows that for every $r \in I \cup E_1$, the subcorrespondence $\left(c, \mathbb{A}^r_{R^{\downarrow}(r)}\right)$ admits an expected utility representation.

Lemma 10. *For any $r \in \Delta(X)$, if $\mathbb{P}^r_{+}$ contains a full-dimensional convex subset of $\Delta(X)$, then there exists a strictly increasing utility function $u_r : X \to \mathbb{R}$, unique up to a positive affine transformation, such that $c(A) = \arg\max_{p \in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^r_{R^{\downarrow}(r)}$.*

Proof. Since $\mathbb{P}^r_{+}$ contains a full-dimensional convex subset of $\Delta(X)$, consider a subset $\mathbb{P} \subseteq \mathbb{P}^r_{+}$ that is a linear transformation of a $|X|-1$ dimensional simplex (hence also full-dimensional and convex). First, notice that for all $p,q \in \mathbb{P}$, we have $\{r,p,q\} \in \mathbb{A}^r_{R^{\downarrow}(r)}$ and $r \notin c(\{r,p,q\})$. Recall that c satisfies WARP and Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$. By letting $p \succsim_r q$ if $p \in c(\{r,p,q\})$, we obtain a binary relation $(\succsim_r, \mathbb{P})$ that is complete, transitive, continuous, and satisfies the standard von Neumann-Morgenstern Independence, and it is well-known that there exists a utility function $u_r : X \to \mathbb{R}$, unique up to a positive affine transformation, such that $c(A) = \arg\max_{p \in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^r_{\mathbb{P}}$. Since $(\succsim_r, \mathbb{P})$ satisfies FOSD (Axiom 3.1), u_r is strictly increasing. We normalize this function to $u_r : X \to [0,1]$ where $u_r(w) = 0$ and $u_b(b) = 1$.

We now show that this utility function can explain $\left(c, \mathbb{A}^r_{R^{\downarrow}(r)}\right)$. First, note that for any two lotteries $p,q \in \Delta(X)$, there exist two (possibly different) lotteries $p',q' \in \mathbb{P}$ such that (p',q') is a $\mathbb{P}$-common mixture of (p,q). This can be done by taking an arbitrary $s \in \text{Int}\,\mathbb{P}$ and α large enough so that both p' and q' enter $\mathbb{P}$ (this is why we need $\mathbb{P}$ to be full-dimensional and convex). Now consider any $p \in R^{\downarrow}(r)$ and let (r',p') be a $\mathbb{P}$-common mixture of (r,p). Since c satisfies Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$, for $i = r,p$, $i' \in c(\{r,r',p'\})$ if and only if $i \in c(\{r,p\})$. Now take any $p,q \in R^{\downarrow}(r)$ such that $p \in c(\{r,p\})$ and $q \in c(\{r,q\})$, then again by Independence over $\mathbb{A}^r_{R^{\downarrow}(r)}$, $p' \in c(\{r,p',q'\})$ if and only if $p \in c(\{r,p,q\})$, where (p',q') is a $\mathbb{P}$-common mixture of (p,q). We have thus shown that $c(\{r,p\}) = \arg\max_{s \in \{r,p\}} \mathbb{E}_s u_r(x)$ for all $\{r,p\} \in \mathbb{A}^r_{R^{\downarrow}(r)}$ and $c(\{r,p,q\}) = \arg\max_{s \in \{r,p,q\}} \mathbb{E}_s u_r(x)$ for all $\{r,p,q\} \in \mathbb{A}^r_{R^{\downarrow}(r)}$ with $p \in c(\{r,p\})$ and $q \in c(\{r,q\})$. Since c satisfies WARP over $\mathbb{A}^r_{R^{\downarrow}(r)}$, showing $c(A) = \arg\max_{p \in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^r_{R^{\downarrow}(r)}$ is straightforward from here. □

Stage 4: Concave transformations when $r_1, r_2 \in I \cup E_1$

Lemma 11. *For any $r_1, r_2 \in I$, if $r_1 R r_2$, then $u_{r_1} = f \circ u_{r_2}$ for some concave and strictly increasing function $f : [0,1] \to [0,1]$.*

Proof. This proof uses Axiom 3.3. Take any $r_1, r_2 \in I$ such that $r_1 R r_2$. Consider the function $\bar{f}$ whose domain is the set of numbers $\{u_{r_2}(x) : x \in X\}$ such that $u_{r_1}(x) = \bar{f} u_{r_2}(x)$. Since u_{r_1} and u_{r_2} are strictly increasing, $\bar{f}$ is strictly increasing in its domain.

We show that if $x_1 < x_2 < x_3$, then $\bar{f}(\alpha u_{r_2}(x_1) + (1-\alpha) u_{r_2}(x_3)) \geq \alpha \bar{f}(u_{r_2}(x_1)) + (1-\alpha) \bar{f}(u_{r_2}(x_3))$ where α solves $\alpha u_{r_2}(x_1) + (1-\alpha) u_{r_2}(x_3) = u_{r_2}(x_2)$. Suppose not, then there exists β, strictly greater than α, such that $\bar{f}(\alpha u_{r_2}(x_1) + (1-\alpha) u_{r_2}(x_3)) < \beta \bar{f}(u_{r_2}(x_1)) + (1-\beta) \bar{f}(u_{r_2}(x_3)) < \alpha \bar{f}(u_{r_2}(x_1)) + (1-\alpha) \bar{f}(u_{r_2}(x_3))$. Consider the lotteries $\delta = \delta_{x_2}$ and $p = (\delta_{x_1})^{\beta}(\delta_{x_3})$. The above equations give $\mathbb{E}_{\delta} u_{r_1}(x) < \mathbb{E}_{p} u_{r_1}(x)$ and $\mathbb{E}_{\delta} u_{r_2}(x) > \mathbb{E}_{p} u_{r_2}(x)$. Let (δ_1, p_1) be a $\mathbb{P}$-common mixture of (δ, p) where $\mathbb{P}$ is a full-dimensional convex subset of $\mathbb{P}^{r_1}_{+r_2}$ if $c(\{r_1, r_2\}) = \{r_2\}$ and of $\mathbb{P}^{r_1}_{+}$ otherwise (Lemma 9 guarantees the existence of $\mathbb{P}$). Let (δ_2, p_2) be a $\mathbb{P}$-common mixture of (δ, p) where $\mathbb{P}$ is a full-dimensional convex subset of $\mathbb{P}^{r_2}_{+}$. Since $\mathbb{E} u_{r_1}$ explains $\left(c, \mathbb{A}^{r_1}_{R^{\downarrow}(r_1)}\right)$ and $\mathbb{E} u_{r_2}$ explains $\left(c, \mathbb{A}^{r_2}_{R^{\downarrow}(r_2)}\right)$, we have $c(\{r_1, \delta_1, p_1\}) = \{p_1\}$ and $c(\{r_2, \delta_2, p_2\}) = \{\delta_2\}$. Now consider $A = \{r_1, r_2, \delta_1, \delta_2, p_1, p_2\}$, which is in $\mathbb{A}^{r_1}_{R^{\downarrow}(r_1)}$, and so $c(A) = \arg\max_{q \in A} \mathbb{E}_q u_{r_1}(x)$. Because we have established $\mathbb{E}_{r_2} u_{r_1}(x) < \mathbb{E}_{p_1} u_{r_1}(x)$, $\mathbb{E}_{r_1} u_{r_1}(x) < \mathbb{E}_{p_1} u_{r_1}(x)$, and $\mathbb{E}_{\delta_i} u_{r_1}(x) < \mathbb{E}_{p_i} u_{r_1}(x)$ for $i = 1, 2$ (the first two inequality are due to the way p_1 was picked), so we know $c(A) \subseteq \{p_1, p_2\}$. But $c(A) \subseteq \{p_1, p_2\}$ and $c(\{r_2, \delta_2, p_2\}) = \{\delta_2\}$ jointly violate Axiom 3.3.

To complete the proof, extend $\bar{f}$ to a concave function $f : [0,1] \to [0,1]$ (for example by connecting points with lines). □

Lemma 12. *For any $r \in E_1 \cup E_2$ and $p \in R^{\downarrow}(r) \setminus \{r\}$, either p first-order stochastically dominates r or r first-order stochastically dominates p.*

Proof. Take $r \in E_1 \cup E_2$ and $p \in R^{\downarrow}(r)$, $p \neq r$. Let $\alpha = r(b)$, then $r(w) = 1 - \alpha$. If $p(b) < \alpha$ and $p(w) < (1-\alpha)$, then r is an extreme spread of p and pRr, so $p \notin R^{\downarrow}(r)$. Furthermore, it is not possible that $p(b) \geq \alpha$ and $p(w) \geq (1-\alpha)$ since $p \neq r$. Hence either $p(b) \geq \alpha$ and $p(w) \leq (1-\alpha)$ with at least one strict inequality, so p first-order stochastically dominates r, or $p(b) \leq \alpha$ and $p(w) \geq (1-\alpha)$ with at least one strict inequality, so r first-order stochastically dominates p. □

Lemma 13. *For any $r_1, r_2 \in I \cup E_1$, if $r_1 R r_2$, then $u_{r_1} = f \circ u_{r_2}$ for some concave and increasing function $f : [0,1] \to [0,1]$.*

Proof. We use the proof in Lemma 11 with the following modifications. When $r_2 \in E_1$, let (δ_1, p_1) be a $\mathbb{P}$-common mixture of (δ, p), where $\mathbb{P}$ is a full-dimensional convex subset of $\mathbb{P}^{r_1}_{+}$. (Before, we let $\mathbb{P}$ be a full-dimensional convex subset of $\mathbb{P}^{r_1}_{+r_2}$ when $c(\{r_1, r_2\}) = \{r_2\}$, but now such a subset may not exist since $r_2 \notin I$). Since $\delta_2, p_2 \in \mathbb{P}^{r_2}_{+}$ and Lemma 12 guarantees δ_2 and p_2 each first-order stochastically dominates r_2, we replace the argument "$\mathbb{E}_{r_2} u_{r_1}(x) < \mathbb{E}_{p_1} u_{r_1}(x)$" with

"$\mathbb{E}_{r_2} u_{r_1}(x) < \mathbb{E}_{p_2} u_{r_1}(x)$." Everything else goes through according to the proof in Lemma 11, giving the desired result. □

Stage 5: Expected utility when $r \in E_2$ and concave transformations by construction

We are left with $r \in E_2$, the alternatives in $\text{conv}(\{\delta_b, \delta_w\})$ that are weakly preferred to everything they reference dominate. The construction of u_r can be partly arbitrary, where the main goal is to make sure they are related by concave transformations to other utility functions.

By definition of E_2, $\mathbb{P}^r_+ \cap I = \emptyset$, so by Lemma 12 and FOSD (Axiom 3.1), r first-order stochastically dominates p for all $p \in R^{\downarrow}(r) \cap I$. For any $p \in R^{\downarrow}(r) \cap \text{conv}(\{\delta_b, \delta_w\})$, FOSD requires the choice $c(\{r, p\})$ to obey first order stochastic dominance. Together, any strictly increasing utility function $u_r : X \to [0,1]$ will accomplish $c(A) = \arg\max_{p \in A} \mathbb{E}_p u_r(x)$ for all $A \in \mathbb{A}^r_{R^{\downarrow}(r)}$.

We now construct u_r so that it is related to other utility functions by concave transformations. For any strictly increasing utility function u_p, consider the object $\rho^p = \left(\rho^p_2, ..., \rho^p_{|X|-1}\right) \in (0,1)^{|X|-2}$ such that for all $i \in \{2, ..., |X|-1\}$,

$$\rho^p_i = \frac{u_p(x_i) - u_p(x_{i-1})}{u_p(x_{i+1}) - u_p(x_{i-1})} \tag{B.1}$$

(so ρ^p_i satisfies $u_p(x_i) = \rho^p_i u_p(x_{i+1}) + (1 - \rho^p_i) u_p(x_{i-1})$). There is a one-to-one relationship between u_p and ρ^p. Also, it is an algebraic exercise to show that $u_p = f \circ u_q$ for some concave and strictly increasing $f : [0,1] \to [0,1]$ if and only if $\rho^p_i \geq \rho^q_i$ for all i.

Fix $r \in E_2$. Let $\rho^r = \left(\inf_{p \in K_r}(\rho^p_2), ..., \inf_{p \in K_r}\left(\rho^p_{|X|-1}\right)\right)$, where $K_r := (I \cup E_1) \cap \{p : pRr\}$, and subsequently construct u_r using Equation B.1, which is possible as long as K_r is nonempty. Note that when $r \notin \{\delta_b, \delta_w\}$, r must be the mean preserving spread of something in I, so $I \cap \{p : pRr\}$ is nonempty, and so K_r is nonempty. In the exception where $r \in \{\delta_b, \delta_w\}$ and K_r is empty, this implies rRp for all $p \in \Delta(X) \setminus \{\delta_b, \delta_w\}$. Then, we let

$$\rho^r_i = \frac{1}{2}(1) + \frac{1}{2} \sup_{p \in \Delta(X) \setminus \{\delta_b, \delta_w\}} \rho^p_i$$

for all i and construct u_r using Equation B.1. For any $p \in \Delta(X) \setminus \{r\}$, this construction results in $\rho^r_i \geq \rho^p_i$ for all i, with equality for p that also falls into this exception (there are at most two of them, δ_b and δ_w).

We now show that for any $r_1, r_2 \in \Delta(X)$ where $r_1 R r_2$, we have $\rho^{r_1}_i \geq \rho^{r_2}_i$ for all i. This is already shown for any $r_1, r_2 \in I \cup E_1$ by Lemma 13. It is also already shown for the special cases in the preceding paragraph, by careful construction. Hence, we restrict attention to the remaining cases. Suppose $\rho^{r_1}_i < \rho^{r_2}_i$ for some i. Then $\inf_{p \in K_{r_1}}(\rho^p_i) < \rho^{r_2}_i$, so $\rho^p_i < \rho^{r_2}_i$ for some $p \in K_{r_1}$. However, since R is transitive, $p \in K_{r_1}$ implies pRr_2; and since $p \in I \cup E_1$, this contradicts Lemma 13. Say $r_1 \in I \cup E_1$, $r_2 \in E_2$, but $\rho^{r_1}_i < \rho^{r_2}_i$ for some i. Then $\rho^{r_1}_i < \inf_{p \in K_{r_2}}(\rho^p_i)$, so $\rho^{r_1}_i < \rho^p_i$ for all $p \in K_{r_2}$. But $r_1 \in K_{r_2}$, a contradiction. Finally, for $r_1, r_2 \in E_2$, either $K_{r_1} = K_{r_2}$ or $K_{r_1} \subsetneq K_{r_2}$. If it is the former, it is immediate that $\rho^{r_1} = \rho^{r_2}$. If it is the later, then $\rho^{r_1}_i = \inf_{p \in K_{r_1}}(\rho^p_i) \geq \inf_{p \in K_{r_2}}(\rho^p_i) = \rho^{r_2}_i$

for all i, as desired.

Thus, we have now shown that for any $r_1, r_2 \in \Delta(X)$ such that $r_1 R r_2$, $\rho_i^{r_1} \geq \rho_i^{r_2}$ for all i, or equivalently $u_{r_1} = f \circ u_{r_2}$ for some concave and strictly increasing $f : [0,1] \to [0,1]$.

B.3 Proof of Theorem 2

"**If**" is straightforward, where compliance with Axiom 4.4 is shown in a footnote. I prove "**only if.**" In **Stage 1**, we show that with Axiom 4.1 and Axiom 4.2, for any time $\tau \in T$, the set of all choice problems such that the earliest payment arrives at time τ can be explained by a nonempty set of Discounted Utility specifications, where a typical element of this set is a utility function and a discount factor. In **Stage 2**, we show that at least one (consumption) utility function u can be supported for all $\tau \in T$, and for each $\tau \in T$ we set as $\hat{\delta}_\tau$ the corresponding discount factor associated with u for τ; this is the more involved portion of the proof and it uses Axiom 4.4. In **Stage 3**, with Axiom 4.3, we show the desired relationship between $\hat{\delta}_\tau$ and $\hat{\delta}_{\tau'}$ for any two τ, τ'. Note that the representation constructed has discount factors indexed by time, not alternatives, so in **Stage 4** we convert them back to alternatives.

Stage 1: DU representation for each $\tau \in T$

By Lemma 1 and Lemma 3, for any $x \in X$ and $\tau \in T$, c satisfies WARP and Stationarity over $S_{(x,\tau)} := \{A \in \mathcal{A} : (x,\tau) \in \Psi(A)\}$ (the additional compliance with WARP is implied by taking $a = 0$ in the definition of Stationarity). In fact, WARP and Stationarity hold even when we consolidate the collection of choice problems where the earliest payment arrives at the same time (although the payments themselves may be different), which we now show. Let $S_{(\cdot,\tau)} := \cup_{x \in X} S_{(x,\tau)}$.

Lemma 14. *For any $\tau \in T$, c satisfies WARP and Stationarity over $S_{(\cdot,\tau)}$.*

Proof. Take any two choice sets $A, B \in S_{(\cdot,\tau)}$. Suppose it is not true that c satisfies WARP or Stationarity over $\{A, B\}$. Therefore, it must be that $\Psi(A) \cap \Psi(B) = \emptyset$. Now let's take the worse payment at τ for each set: $(x^*, \tau) \in A$ such that $x^* \leq x$ for all $(x,\tau) \in A$ and $(y^*, \tau) \in B$ such that $y^* \leq y$ for all $(y,\tau) \in B$. Suppose without loss of generality $x^* < y^*$ (due to $\Psi(A) \cap \Psi(B) = \emptyset$). By Axiom 4.1, adding (x^*, τ) to B would not alter the choice, i.e., $c(B \cup \{(x^*,\tau)\}) = c(B)$. Let $B^* := B \cup \{(x^*,\tau)\}$; note that A and B^* are both in $S_{(x^*,\tau)}$, and therefore c satisfies WARP or Stationarity over $\{A, B^*\}$. If it is Stationarity that is violated between A and B, then it is also violated between A and B^*, a contradiction. If it is WARP that is violated between A and B, it remains to show, due to $(x^*,\tau) \in A$, if $A \subseteq B$ then $A \subseteq B^*$ and there is a contradiction, whereas if $A \supseteq B$ then $A \supseteq B^*$ and there is a contradiction. □

We just established that c satisfies WARP and Stationarity over $S_{(\cdot,\tau)}$. This will give us, from the choices in $\left(c, S_{(\cdot,\tau)}\right)$, a revealed preference relation on $\{(x,t) \in X \times T : t \geq \tau\}$ that is complete, transitive, continuous, and satisfies stationarity, and then it is well-known (Fishburn and Rubinstein

(1982)) that along with Axiom 4.1 we obtain (many) Discounted Utility (DU) representations, for instance by translating the time-index by $-\tau$ so that time τ is, in that instance, time 0.

Stage 2: u_τ can coincide with u_0 for each $\tau \in T$

With existence guaranteed, arbitrarily pick a DU representation with parameters $\left(\hat{\delta}_0, u_0\right)$ that explains $\left(c, S_{(\cdot,0)}\right)$. Define $U_0 : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ by $U_0(x,t) := \hat{\delta}_0^t u_0(x)$. For every $\tau \in (0, \bar{t})$, arbitrarily pick a DU representation $\left(\tilde{\delta}_\tau, \tilde{u}_\tau\right)$ that explains $\left(c, S_{(\cdot,\tau)}\right)$ and define $U_\tau : X \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ by $U_\tau(x,t) := \tilde{\delta}_\tau^t \tilde{u}_\tau(x)$. We proceed to show that for every $\tau \in (0, \bar{t})$, there exists a DU representation $\left(\hat{\delta}_\tau, u_\tau\right)$ that explains $\left(c, S_{(\cdot,\tau)}\right)$ where $u_\tau = u_0$. Fix a τ. This boils down to identifying a certain relationship between U_0 and U_τ due to the fact that they are DU representations and Axiom 4.4—indifferences are preserved under a common delay multiplier λ.

Fact 1. *For any $\tau \in [0, \bar{t})$, $t \geq 0$, and $q \geq 0$, $U_\tau(x,0) = U_\tau(y,t)$ if and only if $U_\tau(x,q) = U_\tau(y, q+t)$.*

Lemma 15. *For any $x \in (a,b)$ (resp. $x = a$ and $x = b$), there exists an open interval $B = (x^-, x^+) \subseteq (a,b)$ (resp. proper interval $B = [a, x_a^+)$ where $x_a^+ < b$ and proper interval $B = (x_b^-, b]$ where $x_b^- > a$) that contains x such that for some unique $\lambda \in \mathbb{R}$, $U_0\left(z_1, \tilde{t}_1\right) = U_0\left(z_2, \tilde{t}_2\right)$ if and only if $U_\tau\left(z_1, \hat{t}_1\right) = U_\tau\left(z_2, \hat{t}_1 + \lambda\left(\tilde{t}_2 - \tilde{t}_1\right)\right)$ for all $z_1, z_2 \in B$.*

Proof. Fix any $x \in (a,b)$. Consider $i \in \{0, \tau\}$. Since $U_i(\cdot,\cdot)$ is continuous and decreasing in it's second argument, there exists $q \in (i, \bar{t})$ such that $c(\{(a,i),(x,q)\}) = \{(x,q)\}$. Since there exists an open interval in $(i, \bar{t})$ that contains q, by continuity of $U_i(\cdot,\cdot)$, there exists an open interval O_i in X that contains x such that $x' \in O_i$ implies $c(\{(a,i),(x,q),(x',q')\}) = \{(x,q),(x',q')\}$ for some $q' \in (i, \bar{t})$. *Observation*: for every $x_1, x_2 \in O_i$ such that $x_1 < x_2$, since we have $c(\{(a,i),(x,q),(x_1,t_1)\}) = \{(x,q),(x_1,t_1)\}$ for some t_1, $c(\{(a,i),(x,q),(x_2,t_2)\}) = \{(x,q),(x_2,t_2)\}$ for some t_2, and Lemma 14, we have $c(\{(x_1,i),(x_2,i+t_2-t_1)\}) = \{(x_1,i),(x_2,i+t_2-t_1)\}$ in $\left(c, S_{(\cdot,i)}\right)$.

Now consider an open interval $(x^-, x^+) \subseteq O_\tau \cap O_0$ that contains x. Consider any $x_1, x_2, z \in (x^-, x^+)$ where $x_1 < z < x_2$. We show an intermediate result that (i) $U_0(x_1,0) = U_0(z, \alpha_z t) = U_0(x_2,t)$ if and only if (ii) $U_\tau(x_1,0) = U_\tau(z, \alpha_z t') = U_\tau(x_2,t')$. Say (i) holds (for some α_z). Due to the *observation*, $x_1, x_2 \in O_0$, and Lemma 14, we have $c(A) = A$ where $A = \{(x_1,0),(z,\alpha_z t),(x_2,t)\}$. Due to the *observation* and $x_1, x_2 \in O_\tau$, we have $c(\{(x_1,\tau),(x_2,\tau+t')\}) = \{(x_1,\tau),(x_2,\tau+t')\}$ for some t'. Consider the choice set $B = \{(x_1,\tau),(z,\tau+\alpha_z t'),(x_2,\tau+t')\}$, and note that B is related to A by transforming the time of each timed payment in A from $\hat{t}$ to $\lambda^* \hat{t} + d^*$, where $\lambda^* = \frac{t'}{t}$ and $d^* = \tau$. Then, invoking Axiom 4.4 gives $c(B) = B$, which gives (ii) as desired. The converse, (ii) implies (i), can be shown analogously. Due to Fact 1, we also note that $U_0(x_1,0) = U_0(z,\alpha_z t) = U_0(x_2,t)$ if and only if $U_\tau(x_1,0) = U_\tau(z,\alpha_z t') = U_\tau(x_2,t')$.

Consider any $z_1, z_2, z_3, z_4 \in (x^-, x^+)$. There exist $x_1, x_2 \in (x^-, x^+)$ such that $z_i \in (x_1, x_2)$ for all i. The intermediate result gives, for all $i, j \in \{1,2,3,4\}$, $U_0(x_1,0) = U_0(z_i, \alpha_i t) = U_0(x_2,t)$ if and only if $U_\tau(x_1,0) = U_\tau(z_i,\alpha_i t') = U_\tau(x_2,t')$, so $U_0(z_i,\alpha_i t) = U_0(z_j,\alpha_j t)$ if and only if

$U_\tau(z_i, \alpha_i t') = U_\tau(z_j, \alpha_j t')$, so by Fact 1, $U_0(z_i, 0) = U_0(z_j, (\alpha_j - \alpha_i)\, t)$ if and only if $U_\tau(z_i, 0) = U_\tau(z_j, (\alpha_j - \alpha_i)\, t')$, which means $U_0(z_i, 0) = U_0\left(z_j, \tilde{t}\right)$ if and only if $U_\tau(z_i, 0) = U_\tau\left(z_j, \lambda\tilde{t}\right)$ where $\lambda = \frac{t'}{t}$. Note that λ is independent of i, j, hence the same λ applies to relate z_1, z_2 and to relate z_3, z_4. Invoking Fact 1 once more completes the proof for the existence of λ. Since $\lambda = \frac{t'}{t}$, where t, t' are the unique solutions to $U_0(x_1, 0) = U_0(x_2, t)$ and $U_\tau(x_1, 0) = U_\tau(x_2, t')$, therefore λ is unique (for the given $x \in (a, b)$).

For $x = a$ and $x = b$, the proof is similar other than we replace open intervals (x^+, x^-) with half-open intervals $[a, x_a^+)$ and $(x_b^-, b]$. □

Lemma 16. *There exists $\lambda \in \mathbb{R}$ such that for all $x^* \in X$, $U_0(a, 0) = U_0(x^*, t^*)$ if and only if $U_\tau(a, 0) = U_\tau(x^*, \lambda t^*)$. Moreover, λ is unique.*

Proof. Let $\mathbb{C} := \left\{[a, x_a^+), (x_b^-, b]\right\} \cup \{(x_x^+, x_x^-) : x \in (a, b)\}$ be the collection intervals guaranteed by Lemma 15. Note that $\mathbb{C}$ is an open cover of the closed and bounded interval $[a, b]$, so a finite subcover $\bar{\mathbb{C}}$ is guaranteed by the Heine–Borel theorem. Consider a finite sequence of intervals in $\bar{\mathbb{C}}$, $(B_k)_{k=1}^K$, such that the first interval is $B_1 = [a, x_a^+)$, last interval is $B_K = (x_b^-, b]$, and for all $k \in \{1, K-1\}$, $B_k \cap B_{k+1} \neq \emptyset$. This is guaranteed by the fact that $\bar{\mathbb{C}}$ is a cover of $[a, b]$ and the intervals in $\bar{\mathbb{C}}$ are open except for $[a, x_a^+)$ and $(x_b^-, b]$. Then, for every two consecutive intervals B_k, B_{k+1}, the unique λ's guaranteed by Lemma 15, one for B_k and another for B_{k+1}, must coincide due to the nondegenerate intersection $B_k \cap B_{k+1}$. Iterating through this finite sequence of intersecting consecutive intervals guarantees, for every $x^* \neq a$, an increasing sequence of payments $(x_k)_{k=1}^M$ such that $x_1 = a$, $x_M = x^*$, and for some λ, $U_0(x_k, 0) = U_0(x_{k+1}, t)$ if and only if $U_\tau(x_k, 0) = U_\tau(x_{k+1}, \lambda t)$ for all $k \in \{1, ..., M-1\}$. The rest is straightforward using Fact 1 (for example if $M = 3$, we have $U_0(a, 0) = U_0(x_1, t_1) = U_0(x^*, t^*)$ if and only if $U_\tau(a, 0) = U_\tau(x_1, \lambda t_1) = U_\tau(x^*, \lambda t_1 + \lambda(t^* - t_1))$, which completes the proof since $\lambda t_1 + \lambda(t^* - t_1) = \lambda t^*$). □

To recover λ, take any $x_1, x_2 \in X$ such that $x_1 < x_2$. For some t and t', $U_0(x_1, 0) = U_0(x_2, t)$ and $U_\tau(x_1, 0) = U_\tau(x_2, t')$. Then since we must have $\lambda t = t'$, we have $\lambda = \frac{t'}{t}$. With Lemma 16, we conclude that $\left(\hat{\delta}_\tau, u_\tau\right)$ where $u_\tau = u_0$ and $\hat{\delta}_\tau = \hat{\delta}_0^{-\lambda}$ is a DU representation for $\left(c, S_{(\cdot,\tau)}\right)$.

The analysis thus far was for $\tau \in (0, \bar{t})$. When $\tau = \bar{t}$, since every choice problem in $S_{(\cdot,\bar{t})}$ contains only timed payments that pay at time $\bar{t}$, a DU representation is trivially established with any positive $\hat{\delta}_{\bar{t}}$ and any strictly increasing $u_{\bar{t}}$. Therefore, we set $u_{\bar{t}} = u_0$ and $\hat{\delta}_{\bar{t}} = \sup_{\tau \in [0,\bar{t})} \hat{\delta}_\tau$ (this is why we cannot guarantee $\hat{\delta}_{\bar{t}} < 1$, even if Axiom 4.1 gives us $\hat{\delta}_\tau \in (0, 1)$ for all τ). From now on, we remove subscript τ from u_τ and simply write u.

Stage 3: $\hat{\delta}_\tau \geq \hat{\delta}_{\tau'}$ for all $\tau > \tau'$

If $\tau = \bar{t}$, this is trivial from the construction of $\hat{\delta}_{\bar{t}}$. Consider any $\tau, \tau' \in [0, \bar{t})$. Continuity of $U_\tau(x, q) = \hat{\delta}_\tau^q u(x)$ and $U_{\tau'}(x, q) = \hat{\delta}_{\tau'}^q u(x)$ guarantee the existence of $y > a$ such that $c(\{(a, \tau), (y, t)\}) = \{(a, \tau), (y, t)\}$ and $c(\{(a, \tau'), (y, t')\}) = \{(a, \tau'), (y, t')\}$ for some $t, t' \in T$, with which we obtain $\hat{\delta}_\tau^\tau u(a) = \hat{\delta}_\tau^t u(y)$ and $\hat{\delta}_{\tau'}^{\tau'} u(a) = \hat{\delta}_{\tau'}^{t'} u(y)$. Note that by Axiom 4.1, $\hat{\delta}_\tau, \hat{\delta}_{\tau'} < 1$,

so $t-\tau>0$ and $t'-\tau'>0$. Suppose for contradiction $\hat{\delta}_{\tau'}>\hat{\delta}_{\tau}$, then $t'-\tau'>t-\tau$, or equivalently $t'>\tau'+t-\tau$. Note also that $\tau'-\tau<0$ implies $\tau'+t-\tau<t$. So $t,t'\in T$ implies $(y,\tau'+t-\tau)\in X\times T$. Putting together what we established, we have $\tau'<\tau<t$, $\tau'<\tau'+t-\tau$, $\hat{\delta}^{\tau}_{\tau'}u(a)<\hat{\delta}^{\tau'}_{\tau'}u(a)=\hat{\delta}^{t'}_{\tau'}u(y)<\hat{\delta}^{\tau'+t-\tau}_{\tau'}u(y)$, and $\hat{\delta}^{t}_{\tau'}u(y)<\hat{\delta}^{\tau'+t-\tau}_{\tau'}u(y)$, which implies $c(\{(a,\tau'),(y,\tau'+t-\tau),(a,\tau),(y,t)\})=\{(y,\tau'+t-\tau)\}$. By Continuity of $U_{\tau}(x,q)$ and $U_{\tau'}(x,q)$, if we consider $y-\epsilon$ for some $\epsilon>0$ sufficiently small, we have $c(\{(a,\tau'),(y-\epsilon,\tau'+t-\tau),(a,\tau),(y-\epsilon,t)\})=\{(y-\epsilon,\tau'+t-\tau)\}$ and $c(\{(a,\tau),(y-\epsilon,t)\})=\{(a,\tau)\}$, which jointly contradict Axiom 4.3 because $\tau'=\tau+d$ and $\tau'+t-\tau=t+d$ where $d=\tau'-\tau>0$.

Stage 4: R **and** $\delta_{(x,t)}$

Create a complete, transitive, and antisymmetric R on Y such that $t<t'$ implies $(x,t)\,R\,(x',t')$, which involves an arbitrary completion between when (x,t) and (x',t') when $t=t'$, and set, for every $(x,t)\in Y$, $\delta_{(x,t)}:=\hat{\delta}_t$.

B.4 Proof of Theorem 3

"**If**" is straightforward. I prove "**only if**." **Stage 1** and **Stage 2** show that with Axiom 5.2 and Axiom 5.1, for each Gini coefficient g, the set of all choice problems where the most balanced alternative has Gini coefficient g can be explained by the maximization of $\tilde{x}+\hat{v}_g(\tilde{y})$ for some unique $\hat{v}_g:[w,+\infty)\to\mathbb{R}$. **Stage 3** shows that $g<g'$ implies $\hat{v}_g(y)-\hat{v}_g(y')\geq\hat{v}_{g'}(y)-\hat{v}_{g'}(y')$ for all $y>y'$. **Stage 4** builds the reference order R using Gini coefficient and arbitrary completion.

Stage 1: $x+v_{(x,y)}(y)$ **for each alternative** $(x,y)\in Y$

Fix $(x,y)\in Y$. Let $R^{\downarrow}((x,y)):=\{(x',y')\in Y:G((x,y))\leq G((x',y'))\}$ and $\mathbb{A}:=\mathbb{A}^{(x,y)}_{R^{\downarrow}((x,y))}=\{A\in\mathcal{A}:(x,y)\in\arg\min_{z\in A}G(z)\}$. By Lemma 3, c satisfies WARP and Quasi-linearity over $\mathbb{A}$ (the additional compliance with WARP is implied by taking $a=0$ in the definition of Quasi-linearity). Let $\mathbb{P}^{(x,y)}:=\left\{(x',y')\in R^{\downarrow}((x,y)):(x',y')\in c(\{(x',y'),(x,y)\})\right\}$.

Since c satisfies WARP over $\mathbb{A}$, by Theorem 4, there exists a utility function $U:Y\to\mathbb{R}$ that explains $(c,\mathbb{A})$.

Note that for all $(x',y')\in R^{\downarrow}((x,y))$, $U(x',y')\geq U(x,y)$ if and only if $(x',y')\in\mathbb{P}^{(x,y)}$. Since c satisfies Quasi-linearity over $\mathbb{A}$ (Axiom 5.2), U restricted to the domain $\mathbb{P}^{(x,y)}$ (which contains (x,y) itself) must be a strictly increasing transformation of $\tilde{x}+v_{(x,y)}(\tilde{y})$ for some unique $v_{(x,y)}:[w,+\infty)\to\mathbb{R}$. Figure B.1 provides an illustration of how $v_{(x,y)}$ is constructed, and $\tilde{x}+v_{(x,y)}(\tilde{y})$ is our target, quasi-linear, utility function. It is straightforward that for all $A\in\mathbb{A}$ such that $A\subseteq\mathbb{P}^{(x,y)}$, the maximization of $\tilde{x}+v_{(x,y)}(\tilde{y})$ gives $c(A)$. Next, we show that this consistency applies to other $A\in\mathbb{A}$. For any $(x',y')\in R^{\downarrow}((x,y))\setminus\mathbb{P}^{(x,y)}$, there is no $A\in\mathbb{A}$ such that $(x',y')\in c(A)$, so we just need to guarantee $x'+v_{(x,y)}(y')<x+v_{(x,y)}(y)$. Suppose for contradiction this inequality fails. Since for some a we have $\{(x+a,y),(x'+a,y')\}\subseteq\mathbb{P}^{(x,y)}$, and

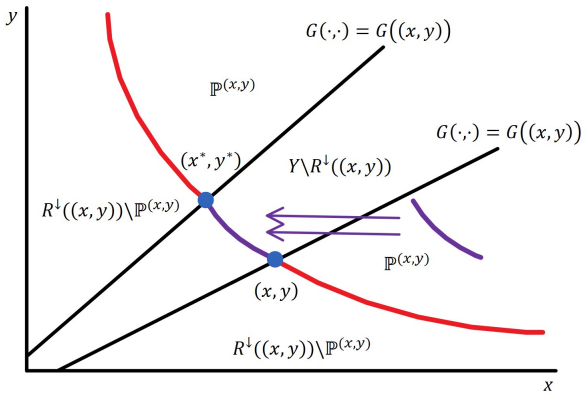

Figure B.1: This figure illustrates the construction of $v_{(x,y)}$ for a fixed $(x,y) \in Y$. The space Y is divided into three regions: (1) Between the two diagonal lines are alternatives in $Y \backslash R^{\downarrow}((x,y))$, they have lower Gini coefficients than (x,y), and therefore they appear in a choice problem where (x,y) is the reference. The alternatives in $R^{\downarrow}((x,y))$ are then split into two groups: (2) those that are chosen when (x,y) is the reference, $\mathbb{P}^{(x,y)}$, and (3) those that are not, $R^{\downarrow}((x,y)) \backslash \mathbb{P}^{(x,y)}$. These two groups are separated by the indifference curve passing through (x,y), the red curve, which partially constructs $v_{(x,y)}$ (partial because Gini coefficient truncates the space). The rest of $v_{(x,y)}$ can be constructed by using an indifference curve that connects $(x+a,y)$ and (x^*+a,y^*), the purple curve, where $G((x,y)) = G((x^*,y^*))$, $c(\{(x,y),(x^*,y^*)\}) = \{(x,y),(x^*,y^*)\}$, and $(x+a,y),(x^*+a,y^*) \in \mathbb{P}^{(x,y)}$.

therefore $(x'+a,y') \in c\{(x,y),(x+a,y),(x'+a,y')\}$, the fact that $\{(x,y),(x+a,y),(x'+a,y')\}$ and $\{(x,y),(x',y')\}$ are both in $\mathbb{A}$ but $(x',y') \notin c(\{(x,y),(x',y')\})$ (because $(x',y') \notin \mathbb{P}^{(x,y)}$) contradicts c satisfies Quasi-linearity over $\mathbb{A}$. It remains to consider the consistency of $\tilde{x} + v_{(x,y)}(\tilde{y})$ for alternative $(x',y') \notin R^{\downarrow}((x,y))$, but this is immediate since there is no $A \in \mathbb{A}$ such that $(x',y') \in \mathbb{A}$. So $\tilde{x} + v_{(x,y)}(\tilde{y})$ explains $(c,\mathbb{A})$.

Stage 2: $x + \hat{v}_g(y)$ for each Gini coefficient g

Fix $g \in [0, 0.5)$, we now show that $v_{(x,y)}$ must coincide for all (x,y) where $G((x,y)) = g$. Consider the collection of choice sets $\mathcal{S} := \{A \in \mathcal{A} : \min_{z \in A} G(z) = g\}$. It turns out that c satisfies WARP and Quasi-linearity over $\mathcal{S}$. To see this, take any two choice problems A_1, A_2 in $\mathcal{S}$. For each $i = 1,2$, there must be an alternative $(x_i,y_i) \in A_i$ such that $G((x_i,y_i)) = g$ and $G((x',y')) \geq g$ for all other (x',y') in A_i. Consider an income distribution (x^*,y^*) such that $x^* \leq \min\{x_1,x_2\}$ and $y^* \leq \min\{y_1,y_2\}$ and $G((x^*,y^*)) = g$. Due to $(x_i,y_i) \in \Psi(A_i \cup \{(x^*,y^*)\})$, Axiom 5.2, and Axiom 5.1, we have $c(A_i) = c(A_i \cup \{(x^*,y^*)\})$ for $i = 1,2$. But $(x^*,y^*) \in \Psi(A_1 \cup A_2 \cup \{(x^*,y^*)\})$, so by Axiom 5.2 again, between $c(A_1 \cup \{(x^*,y^*)\})$ and $c(A_2 \cup \{(x^*,y^*)\})$, which as established are equal to $c(A_1)$ and $c(A_2)$ respectively, WARP and Quasi-linearity must hold. Since c satisfies WARP and Quasi-linearity over $\mathcal{S}$, there is a unique $\hat{v}_g : [w,+\infty) \to \mathbb{R}$ such that the utility function $\tilde{x} + \hat{v}_g(\tilde{y})$ explains $(c,\mathcal{S})$. But every $v_{(x,y)}$ constructed in Stage 1 is also unique, and $\mathbb{A}^{(x,y)}_{R^{\downarrow}((x,y))} \subseteq \mathcal{S}$ if $G((x,y)) = g$, so $v_{(x,y)}$ must coincide for all (x,y) such that $G((x,y)) = g$.

Stage 3: $g < g'$ **implies** $\hat{v}_g(y) - \hat{v}_g(y') \geq \hat{v}_{g'}(y) - \hat{v}_{g'}(y')$ **for all** $y > y'$

Finally we show that the constructed $\hat{v}_g(y)'s$ are systematically related. Consider any $g, g' \in [0, 0.5)$ such that $g < g'$ (reminder: lower g implies greater attainable equality) and any $y, y' \in \mathbb{R}_{\geq 0}$ such that $y > y'$. Define $\bar{v}_g := \hat{v}_g(y) - \hat{v}_g(y')$ and $\bar{v}_{g'} := \hat{v}_{g'}(y) - \hat{v}_{g'}(y')$. We want to show $\bar{v}_g \geq \bar{v}_{g'}$. Suppose not, our goal is to find a contradiction of Axiom 5.3 in choice behavior.

Let z be a number such that $\bar{v}_g < z < \bar{v}_{g'}$. Consider $(x_{g'}, w), (x_g, w) \in Y$ such that $G((x_{g'}, w)) = g'$ and $G((x_g, w)) = g$, which exist because $G((\tilde{x}, w))$ is continuous and increasing in $\tilde{x}$ from $G((w, w)) = 0$ to $\lim_{x \to +\infty} G((x, w)) = 0.5$ and $g, g' \in [0, 0.5)$. Consider $x := z + \Delta$, $x' := 2z + \Delta$ for some $\Delta > 0$ such that $g' \leq \min(\{G((x, y)), G((x', y'))\})$ and $x' > x > \max(\{x_{g'}, x_g\})$, where Δ exists because for any fixed $\bar{y}$, $G((\tilde{x}, \bar{y}))$ is asymptotically increasing in $\tilde{x}$ and $\lim_{x \to +\infty} G((x, \bar{y})) = 0.5$, and $g' \in [0, 0.5)$. Essentially, we have introduced reference points $(x_{g'}, w), (x_g, w)$ that will not be chosen (due in part to Axiom 5.1), forcing the choice to be between (x, y) and (x', y').

We now use the constructed alternatives, $(x, y), (x', y'), (x_{g'}, w), (x_g, w)$, to demonstrate a violation of Axiom 5.3. For the choice problem $\{(x, y), (x', y'), (x_{g'}, w)\}$, $(x_{g'}, w)$ is the reference (so $\hat{v}_{g'}$ is used) and cannot be chosen. Since $\bar{v}_{g'} > z$, or equivalently $z + \hat{v}_{g'}(y) > 2z + \hat{v}_{g'}(y')$, we have $x + \hat{v}_{g'}(y) > x' + \hat{v}_{g'}(y')$, and therefore

$$c(\{(x, y), (x', y'), (x_{g'}, w)\}) = \{(x, y)\}. \tag{B.2}$$

By analogous arguments, $z > \bar{v}_g$ gives $c(\{(x, y), (x', y'), (x_g, w)\}) = \{(x', y')\}$ ($\hat{v}_g$ is used), which also gives

$$c(\{(x, y), (x', y'), (x_{g'}, w), (x_g, w)\}) = \{(x', y')\}. \tag{B.3}$$

due to Axiom 5.2 and $G((x_g, w)) = g \leq g' = G((x_{g'}, w))$. Since $y > y'$, Equation B.2 and Equation B.3 jointly contradict Axiom 5.3.

Stage 4: R **on** Y

Create a complete, transitive, and antisymmetric R on Y such that $G((x, y)) < G((x', y'))$ implies $(x, y) R (x', y')$, which involves an arbitrary completion when $G((x, y)) = G((x', y'))$.